ANALYSIS OF GENERAL MOTORS OIL TEST DATA

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EXECUTIVE SUMMARY

This report presents results from an analysis of the oil test data provided by General Motors. These data consist of 42 observations on four variables: PVIS (percent viscosity increase), WPD (weighted piston deposits), P_reten (phosphorous retention), and New_WPD (adjusted weighted piston deposits). These observations were collected on three oils and five labs. Two of these labs had two test stands each, and the other three had one test stand each. The main goals of the analysis were (1) to calculate the means of the four variables and (2) to estimate the standard deviations of the four variables, taking into account reproducibility (lab-to-lab variability) and repeatability (variability between replicates).

Exploratory analysis identified potentially problematic features of the data. The variable PVIS was found to have a non-normal distribution, a problem that was addressed by transforming this variable into its natural logarithm, log(PVIS), and using this transformed variable in the rest of the analysis. Another potential problem was the presence of an extreme outlier, observation 41, which has extreme values on log(PVIS), WPD, and P_reten. However, after conducting further analysis both with and without this observation, it was found that its presence in the data did not significantly affect the results.

The main findings of the analysis are reported in Tables 1 and 2. Table 1 presents the estimates of the means of each variable by oil type. For each variable, common, rather than individual, means are reported for two of the oils. A common mean was estimated for two oils when the difference of their individual means was found not to be statistically significant. Table 2 presents the estimates of the standard deviation of Oil for each variable as well as estimates of reproducibility and repeatability.

Table 1: Mean Estimates

<table>
<tr><th>Oil</th><th>log(PVIS)</th><th>WPD</th><th>P_reten</th><th>New_WPD</th></tr>
<tr><td>434-2</td><td rowspan="2">4.372330366</td><td rowspan="2">6.818214286</td><td>73.80928571</td><td>5.87214286</td></tr>
<tr><td>GMOD01</td><td rowspan="2">82.35214286</td><td rowspan="2">5.01392857</td></tr>
<tr><td>GMOD02</td><td>4.060780895</td><td>6.185714286</td></tr>
</table>

Table 2: Standard Deviation, Reproducibility, and Repeatability

	log(PVIS)	WPD	P_reten	New_WPD
STD(Oil)	0.17555	0.38936	4.91935	0.53114
Reproducibility	0.82382	1.07279	7.68313	1.47325
Repeatability	0.67719	1.19416	6.33616	1.60520

1. INTRODUCTION

The data analyzed in this report consist of 42 observations on four variables: PVIS (percent viscosity increase), WPD (weighted piston deposits), P_reten (phosphorous retention), and New_WPD (adjusted weighted piston deposits). These observations were collected on three oils and five labs; two of these labs had two test stands each, and the other three had one test stand each. The analysis has two main goals: (1) to calculate the means of the four variables and (2) to estimate the standard deviations of the four variables, taking into account reproducibility (lab-to-lab variability) and repeatability (variability between replicates).

The first section of the report conducted an exploratory data analysis in order to uncover any features of the data that might pose problems for the main analysis. The exploratory analysis found that the variable PVIS had a non-normal distribution – a problem that was dealt with by transforming this variable into its natural logarithm, log(PVIS), and using the transformed variable in the subsequent analysis. The exploratory analysis also found an outlying observation that had extreme values on log(PVIS), WPD, and P_reten. However, this observation was ultimately found not to be a problem since analyses in subsequent sections run both with and without this observation produced essentially the same results.

The second section of the report estimated the means of log(PVIS), WPD, P_reten, and New_WPD. The approach was to use four separate fixed effects analysis of variance (ANOVA) models, one for each variable, each of which treated Oil, Lab, and Stand as factors that potentially influenced the variable in question. For each variable, the model as a whole and Oil in particular were found to be statistically significant. Four sets of multiple comparisons were then conducted to determine how the mean values of each variable differed by oil type. The results of these multiple comparisons thus made it possible to estimate the means of log(PVIS), WPD, P_reten, and New_WPD for each type of oil.

The third section of the report estimated the variance components of log(PVIS), WPD, P_reten, and New_WPD and uses these estimates to calculate the standard deviation as well as reproducibility and repeatability for each variable. As in Section 2, this section relies on four ANOVA models, one for each variable. However, whereas a fixed effects model was used to estimate the means, a random effects model was used to estimate variance components. The use of a random effects model was necessary because a fixed effects model cannot be used to estimate variance components – and, similarly, a random effects model cannot be used to estimate the means.

2. EXPLORATORY DATA ANALYSIS

This section examines the distributions of the four response variables, PVIS, WPD, P_reten, and New_WPD with the goal of determining whether they have any problematic features that might affect the subsequent analysis. Figures 1, 2, and 3 contain, respectively, a histogram, boxplot, and normal probability plot for PVIS. The histogram is right-skewed indicating potential non-normality of the distribution. The boxplot indicates the presence of two potential outliers. These outliers have values of 174.6 and 148.9, which are 3.5 and 2.6 standard deviations, respectively, from the mean of 75.77. The probability plot has a curved pattern, which, as with the histogram, indicates that the data are non-normal. More formally, the p-value of 0.01 for the Anderson-Darling test (provided in figure 3) indicates rejection of the hypothesis of normality.

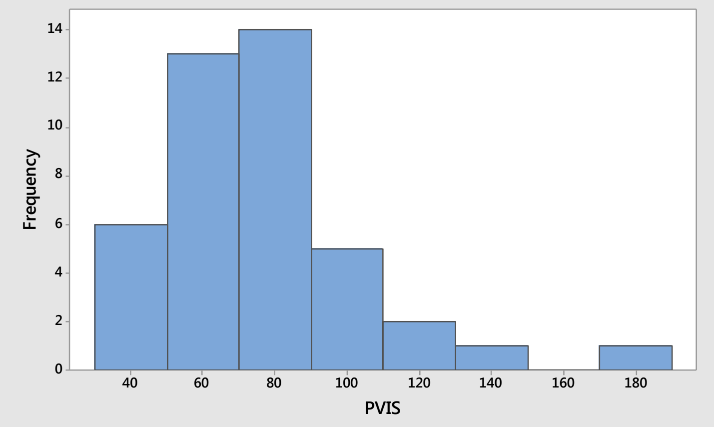


Figure 1: Histogram of PVIS

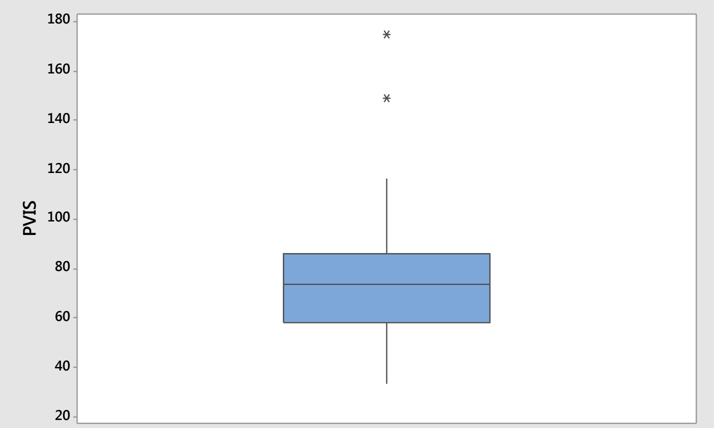


Figure 2: Boxplot of PVIS

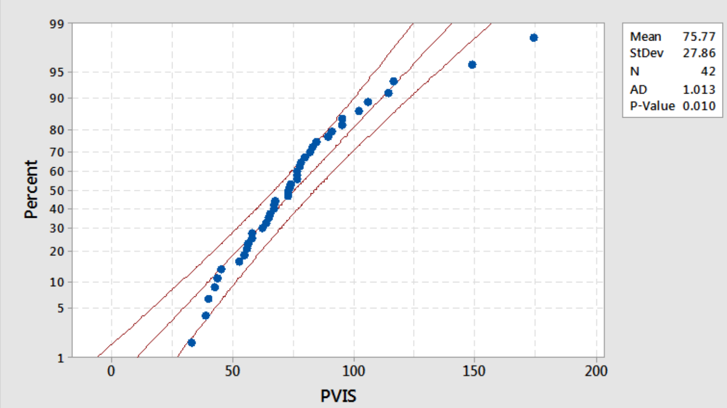
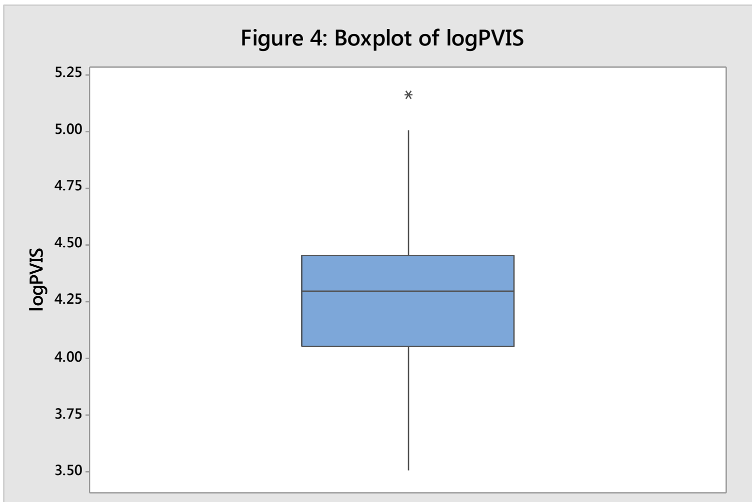
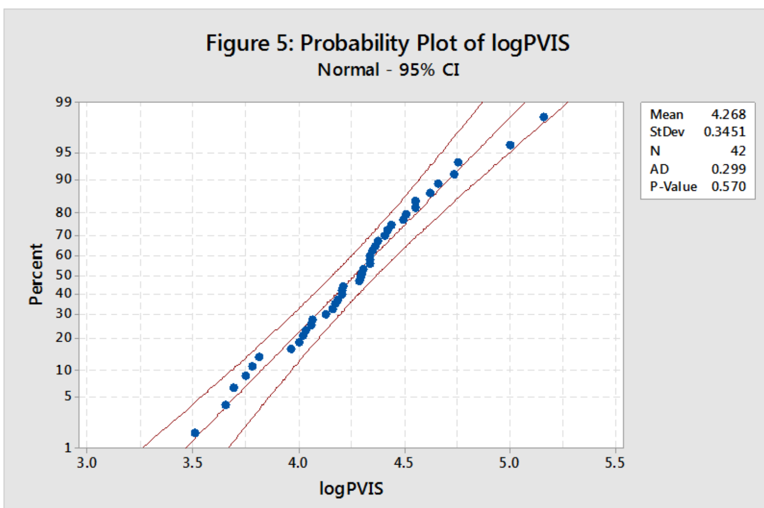


Figure 3: Probability Plot of PVIS
Normal - 95% CI

The non-normality of PVIS could potentially be addressed through a natural-logarithm transformation. The boxplot in Figure 4 indicates the presence of only one outlier once the variable is transformed into its natural logarithm. More importantly, the normal probability plot in Figure 5 shows that the natural logarithm of PVIS has a distribution much closer to normal than does the original variable. Given these results, the analyses that follow will focus on log(PVIS) rather than PVIS.





The variable WPD does not appear to have any of the problematic features of PVIS. Its histogram, boxplot, and normal probability plot, presented in Figures 6, 7, and 8, respectively, do not indicate that the distribution of WPD deviates seriously from normality, nor do they indicate the presence of significant outliers.

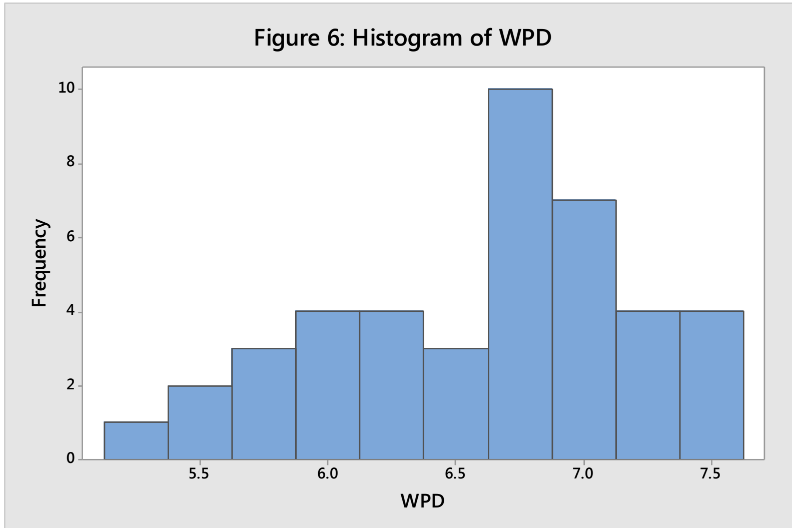


Figure 6: Histogram of WPD
Frequency
10
8
6
4
2
0
5.5
6.0
6.5
7.0
7.5
WPD

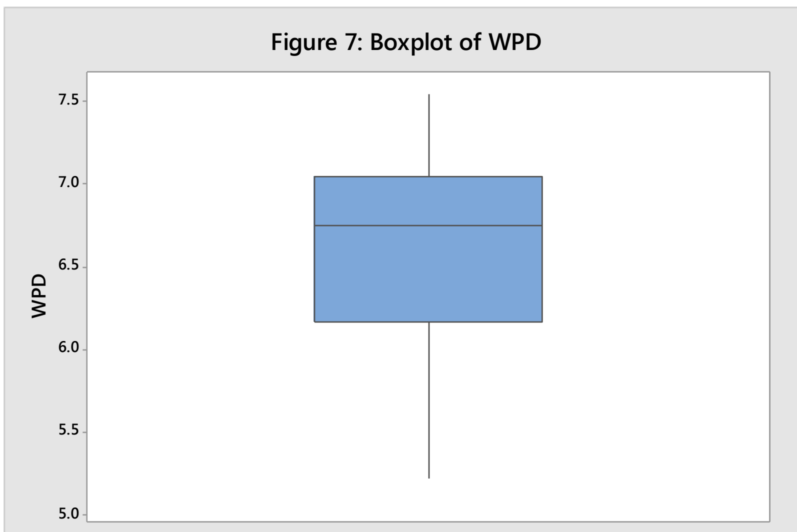


Figure 7: Boxplot of WPD
WPD
7.5
7.0
6.5
6.0
5.5
5.0

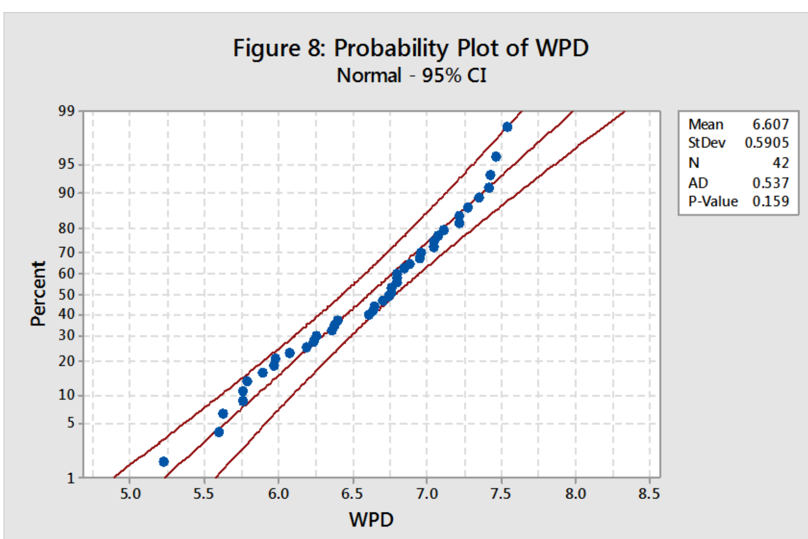
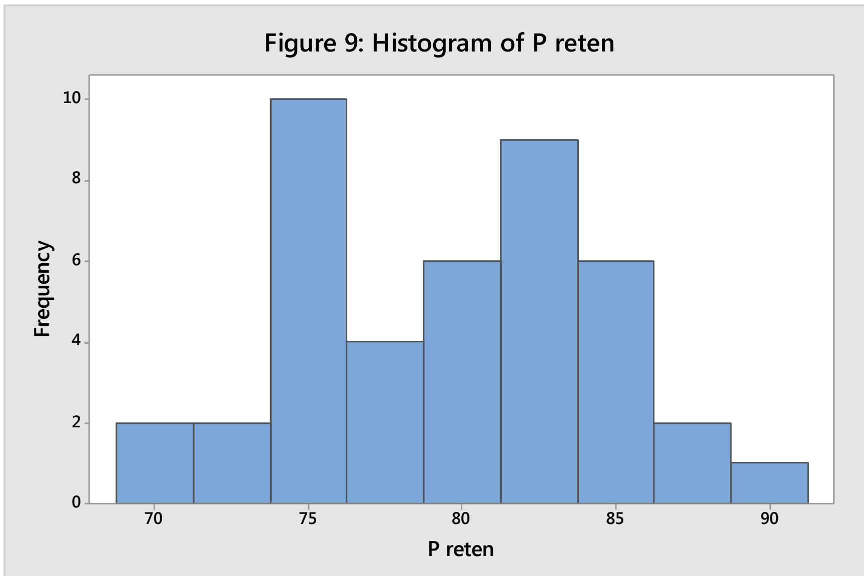
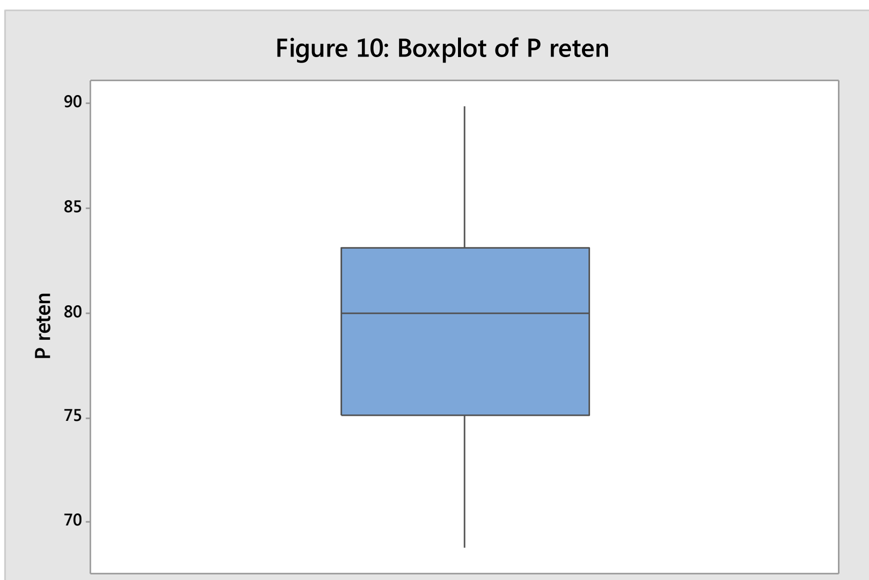
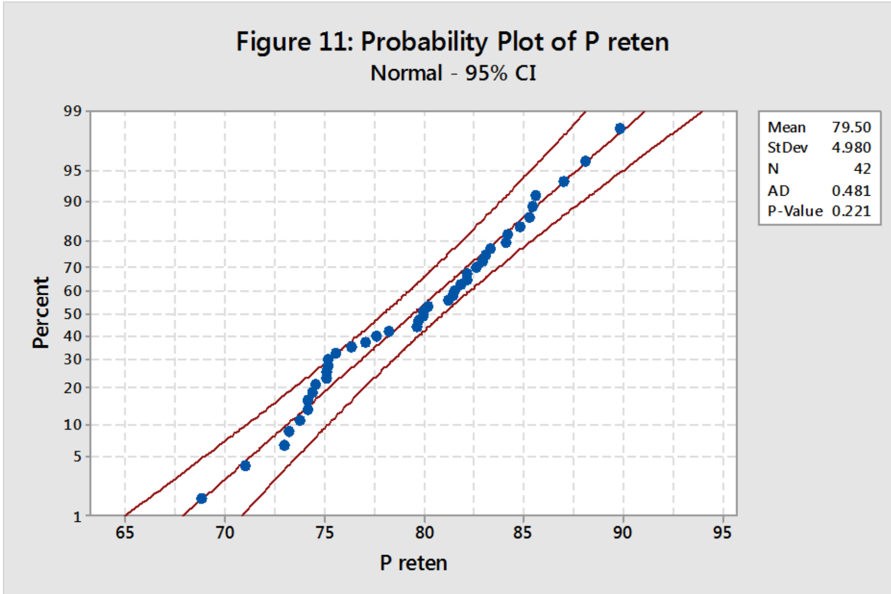


Figure 8: Probability Plot of WPD
Normal - 95% CI
Percent
99
95
90
80
70
60
50
40
30
20
10
5
1
5.0
5.5
6.0
6.5
7.0
7.5
8.0
8.5
WPD
Mean 6.607
StDev 0.5905
N 42
AD 0.537
P-Value 0.159

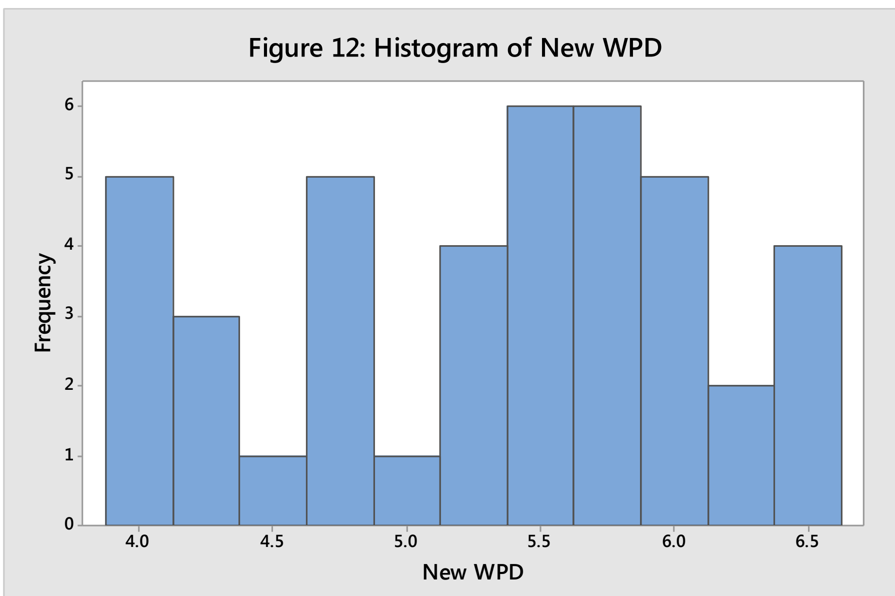
The histogram in Figure 9 reveals that the distribution of P_reten takes a bimodal form. While this potentially bimodal feature is not in itself problematic, it is noteworthy that it is the result of the effect of the variable Oil on P_reten. In particular, the first peak in Figure 9 represents the mode for oil 434-2, while the second peak represents the combined mode of oils GMOD01 and GMOD02. The boxplot and normal probability plot for P_reten, as seen in Figures 10 and 11, indicate, respectively, that this variable has no major outliers and has a normal distribution.

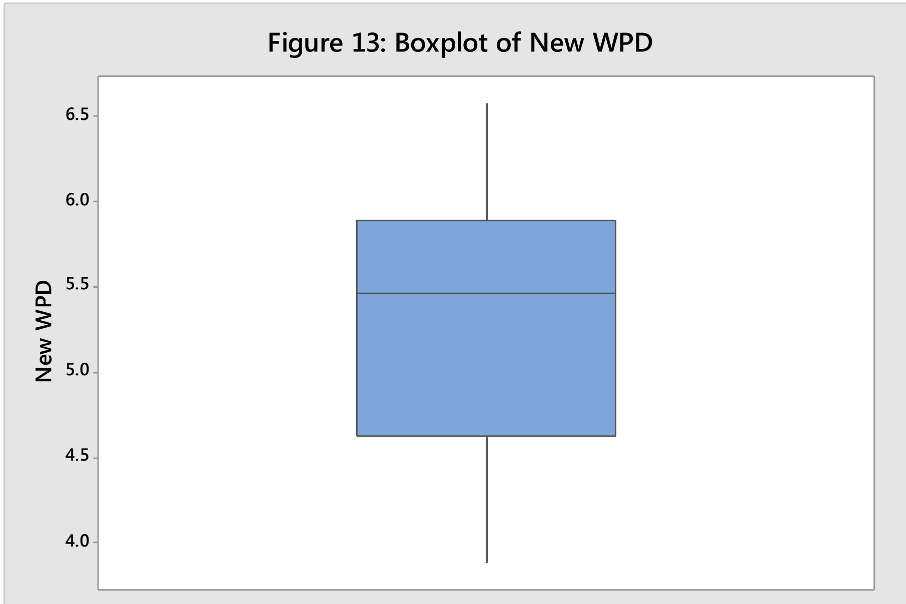


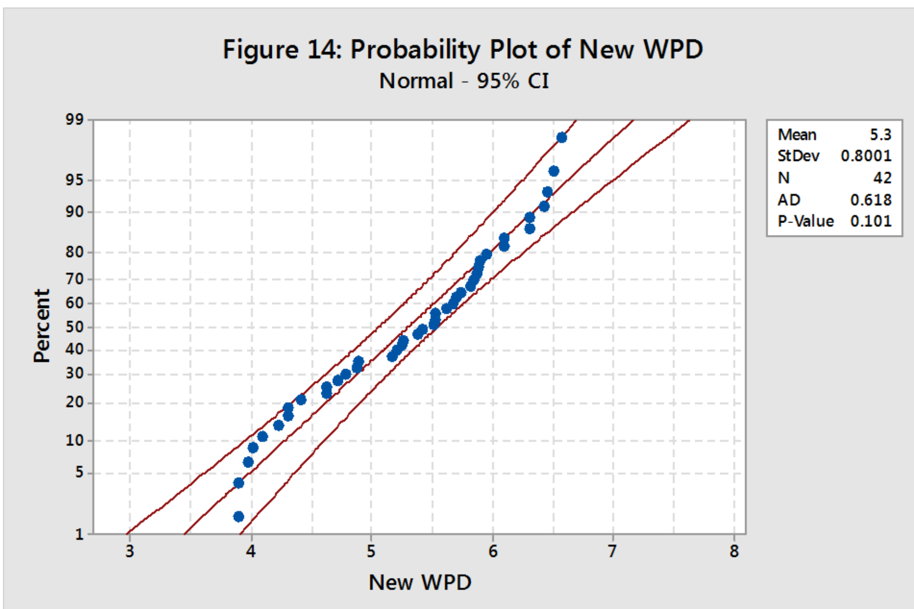




The histogram for New_WPD, presented in figure 12, has a somewhat irregular shape, but, given the relatively small number of observations in the dataset, the pattern in the histogram is not in itself a cause for concern. The boxplot in Figure 13 shows that New_WPD has no major outliers. Figure 14 indicates that New_WPD's distribution deviates slightly from normality, but the Anderson-Darling *p*-value of 0.101 indicates that the level of deviation is not extreme and is within acceptable limits.







Finally, further analysis of the outlier identified in Figure 4, the boxplot for log(PVIS), indicates that this observation, which is observation 41 in the data set, also has extreme values on WPD and P_reten. Not only does it have the highest value of log(PVIS) (5.16, 2.6 standard deviations above the mean), but it also has the lowest value of WPD (5.22, 2.4 standard deviations below the mean) and the highest value of P_reten (89.85, 2.1 standard deviations above the mean). Since this observation was obtained from operationally valid tests, it will not be excluded from the analysis. However, in order to assess its impact on the results, the analyses that follow are carried out both with and without this observation.

3. ESTIMATING THE MEAN

This section uses ANOVA to find estimates of the mean values of log(PVIS), WPD, P_reten, and New_WPD. Each response variable was analyzed individually using a multi-way ANOVA. The following model was used for each of the four variables:

$$y_{ijkl} = \mu + OIL_i + LAB_j + STAND_{k(j)} + \varepsilon_{(ijk)l}$$

$$i = 1,2,3$$

$$j = 1,2,3,4,5$$

k =1,2, for labs A and G

k =1, for labs B, D, and F

l =1,2

$$\varepsilon_{(ijk)l} \sim N(0, \sigma^2)\ i.i.d.,$$

where the subscript l represents the replicates for each combination of Oil, Lab, and Stand and "i.i.d." stands for "independently and identically distributed." In this model, the variables Oil, Lab, and Stand were all treated as fixed factors, and Stand was nested in Lab. Stand was treated as a nested variable because the stands are specific to the labs in which they are located and, thus, cannot be treated as identical across labs. No interaction terms were included in any of the three models as initial investigations showed them to be insignificant.

In the following presentation, the focus is on determining (1) whether the model as a whole is significant for log(PVIS), WPD, P_reten, and New_WPD and (2) whether the variable Oil in particular has a significant effect on the mean value of each variable. If Oil does have an effect, the means of the oil types which are significantly different from one another will be estimated separately. An analysis of how the mean value of each variable differs among the different labs can be found in Appendix I. Further, a detailed examination of the validity of the assumptions underlying the model for each of the variables can be found in Appendix II.

Table 3 presents the results from running the model with log(PVIS) as the response variable. The model is highly significant with a p-value of < 0.0001, and the p-values for Oil, Lab, and Stand are also significant at the $\alpha = 0.05$ level. The analysis was also run with the outlying observation (41) excluded, and the results obtained were essentially the same as those presented in Table 3.

Table 3: ANOVA Results for log(PVIS)

Dependent Variable: LogPVIS

Source	DF	Sum of Squares	Mean Square	F Value	Pr > F
Model	8	2.91413832	0.36426729	6.10	<.0001
Error	33	1.96966589	0.05968685		
Corrected Total	41	4.88380421			

R-Square	Coeff Var	Root MSE	LogPVIS Mean
0.596694	5.723557	0.244309	4.268481

Source	DF	Type I SS	Mean Square	F Value	Pr > F
Oil	2	0.98224787	0.49112394	8.23	0.0013
Lab	4	1.33925295	0.33481324	5.61	0.0015
Stand(Lab)	2	0.59263750	0.29631875	4.96	0.0130

Source	DF	Type III SS	Mean Square	F Value	Pr > F
Oil	2	0.98224787	0.49112394	8.23	0.0013
Lab	4	1.33925295	0.33481324	5.61	0.0015
Stand(Lab)	2	0.59263750	0.29631875	4.96	0.0130

Multiple comparisons were then conducted to investigate more precisely how the mean levels of log(PVIS) differ among the different types of oil. As shown in Table 4, the results indicate that oil GMOD02 has a lower mean log(PVIS) than both of the other two oils and that oils 434-2 and GMOD01 have mean values of log(PVIS) that are not significantly different from one another.

Table 4: Comparison of log(PVIS) Means by Oil

Tukey's Studentized Range (HSD) Test for LogPVIS

Note: This test controls the Type I experimentwise error rate, but it generally has a higher Type II error rate than REGWQ.

Alpha	0.05
Error Degrees of Freedom	33
Error Mean Square	0.059687
Critical Value of Studentized Range	3.47019
Minimum Significant Difference	0.2266

Means with the same letter are not significantly different.

Tukey Grouping	Mean	N	Oil
A	4.42454	14	434-2
A			
A	4.32012	14	GMOD01
B	4.06078	14	GMOD02

Based on the findings in Table 4, Table 5 presents mean estimates for the different types of oil. Since the multiple comparison found oils 434-2 and GMOD01 not to have significantly different means, a common mean value of log(PVIS) was estimated for these two oils.

Table 5: Mean Estimates for log(PVIS)

Oil	Mean
434-2 & GMOD01	4.372330366
GMOD02	4.060780895

Table 6 presents the results from running the model with WPD as the response variable. The model as a whole is significant with a p-value of 0.0004, and the main effects for both Oil and Lab are significant with p-values of 0.0001 and 0.0086, respectively, while the effect for Stand is highly insignificant with a p-value of 0.8943. The analysis on WPD was also run with the outlying observation (41) excluded, and the results were essentially the same as those presented in Table 6.

Table 6: ANOVA Results for WPD

Dependent Variable: WPD

Source	DF	Sum of Squares	Mean Square	F Value	Pr > F
Model	8	7.84176190	0.98022024	5.01	0.0004
Error	33	6.45225000	0.19552273		
Corrected Total	41	14.29401190			

R-Square	Coeff Var	Root MSE	WPD Mean
0.548605	6.692206	0.442180	6.607381

Source	DF	Type I SS	Mean Square	F Value	Pr > F
Oil	2	4.61603333	2.30801667	11.80	0.0001
Lab	4	3.18188690	0.79547173	4.07	0.0086
Stand(Lab)	2	0.04384167	0.02192083	0.11	0.8943

Source	DF	Type III SS	Mean Square	F Value	Pr > F
Oil	2	4.61603333	2.30801667	11.80	0.0001
Lab	4	3.18188690	0.79547173	4.07	0.0086
Stand(Lab)	2	0.04384167	0.02192083	0.11	0.8943

Multiple comparisons were then conducted to investigate more precisely how the mean levels of WPD differed among the different types of oil. As shown in Table 7, the results indicate that oil GMOD02 has a lower mean WPD value than both of the other two oils and that oils 434-2 and GMOD01 have mean values of WPD that are not significantly different from one another. The results of this multiple comparison procedure are unchanged if the insignificant factor, Stand, is left out of the model.

Table 7: Comparison of WPD Means by Oil

Tukey's Studentized Range (HSD) Test for WPD

Note: This test controls the Type I experimentwise error rate, but it generally has a higher Type II error rate than REGWQ.

Alpha	0.05
Error Degrees of Freedom	33
Error Mean Square	0.195523
Critical Value of Studentized Range	3.47019
Minimum Significant Difference	0.4101

Means with the same letter are not significantly different.

Tukey Grouping	Mean	N	Oil
A	6.9957	14	434-2
A			
A	6.6407	14	GMOD01
B	6.1857	14	GMOD02

Based on the findings in Table 7, Table 8 presents mean estimates for the different types of oil. Since the multiple comparison found oils 434-2 and GMOD01 not to have significantly different means, a common mean value of WPD was estimated for these two oils.

Table 8: Mean Estimates for WPD

Oil	Mean
434-2 & GMOD01	6.818214286
GMOD02	6.185714286

Table 9 presents the results from running the model with P_reten as the response variable. The model as a whole is significant with a *p*-value of < 0.0001, and the main effects for both Oil and Lab are significant with *p*-values of < 0.0001 and 0.0004, respectively, while the effect for Stand is insignificant with a *p*-value of 0.5631. The analysis of P_reten was also run with the outlying observation (41) excluded, and the results were essentially the same as those presented in Table 9.

Table 9: ANOVA Results for P_reten

Dependent Variable: P_reten

Source	DF	Sum of Squares	Mean Square	F Value	Pr > F
Model	8	840.332990	105.041624	19.63	<.0001
Error	33	176.629050	5.352395		
Corrected Total	41	1016.962040			

R-Square	Coeff Var	Root MSE	P_reten Mean
0.826317	2.909928	2.313524	79.50452

Source	DF	Type I SS	Mean Square	F Value	Pr > F
Oil	2	688.0508333	344.0254167	64.28	<.0001
Lab	4	146.0258905	36.5064726	6.82	0.0004
Stand(Lab)	2	6.2562667	3.1281333	0.58	0.5631

Source	DF	Type III SS	Mean Square	F Value	Pr > F
Oil	2	688.0508333	344.0254167	64.28	<.0001
Lab	4	146.0258905	36.5064726	6.82	0.0004
Stand(Lab)	2	6.2562667	3.1281333	0.58	0.5631

Multiple comparisons were then conducted to investigate more precisely how the mean levels of P_reten differed among the different types of oil. As shown in Table 10, the results indicate that oil 434-2 has a lower mean P_reten value than both of the other two oils and that oils GMOD01 and GMOD02 have mean values of P_reten that are not significantly different from one another. The results of this multiple comparison procedure are unchanged if the insignificant factor, Stand, is left out of the model.

Table 10: Comparison of P_reten Means by Oil

Tukey's Studentized Range (HSD) Test for P_reten

Note: This test controls the Type I experimentwise error rate, but it generally has a higher Type II error rate than REGWQ.

Alpha	0.05
Error Degrees of Freedom	33
Error Mean Square	5.352395
Critical Value of Studentized Range	3.47019
Minimum Significant Difference	2.1457

Means with the same letter are not significantly different.

Tukey Grouping	Mean	N	Oil
A	82.8486	14	GMOD02
A			
A	81.8557	14	GMOD01
B	73.8093	14	434-2

Based on the findings in Table 10, Table 11 presents mean estimates for the different types of oil. Since the multiple comparison found oils GMOD01 and GMOD02 not to have significantly different means, a common mean value of P_reten is estimated for these two oils.

Table 11: Mean Estimates for P_reten

Oil	Mean
434-2	73.80928571
GMOD01 & GMOD02	82.35214286

Table 12 presents the results from running the model with New_WPD as the response variable. The model as a whole is significant with a p-value of 0.0003, and the main effects for both Oil and Lab are significant with p-values of 0.0001 and 0.0074, respectively, while the effect for Stand is insignificant with a p-value of 0.9084.

Table 12: ANOVA Results for New_WPD

Dependent Variable: New_WPD

Source	DF	Sum of Squares	Mean Square	F Value	Pr > F
Model	8	14.57999524	1.82249940	5.15	0.0003
Error	33	11.66960476	0.35362439		
Corrected Total	41	26.24960000			

R-Square	Coeff Var	Root MSE	New_WPD Mean
0.555437	11.22006	0.594663	5.300000

Source	DF	Type I SS	Mean Square	F Value	Pr > F
Oil	2	8.56972857	4.28486429	12.12	0.0001
Lab	4	5.94210000	1.48552500	4.20	0.0074
Stand(Lab)	2	0.06816667	0.03408333	0.10	0.9084

Source	DF	Type III SS	Mean Square	F Value	Pr > F
Oil	2	8.56972857	4.28486429	12.12	0.0001
Lab	4	5.94210000	1.48552500	4.20	0.0074
Stand(Lab)	2	0.06816667	0.03408333	0.10	0.9084

Multiple comparisons were then conducted to investigate more precisely how the mean levels of New_WPD differed among the different types of oil. As shown in Table 13, the results indicate that oil 434-2 has a higher mean New_WPD value than both of the other two oils and that oils GMOD01 and GMOD02 have mean values of New_WPD that are not significantly different from one another. The results of this multiple comparison procedure are unchanged if the insignificant factor, Stand, is left out of the model.

Table 13: Comparison of New_WPD Means by Oil

Tukey's Studentized Range (HSD) Test for New_WPD

Note: This test controls the Type I experimentwise error rate, but it generally has a higher Type II error rate than REGWQ.

Alpha	0.05
Error Degrees of Freedom	33
Error Mean Square	0.353624
Critical Value of Studentized Range	3.47019
Minimum Significant Difference	0.5515

Means with the same letter are not significantly different.

Tukey Grouping	Mean	N	Oil
A	5.8721	14	434-2
B	5.2600	14	GMOD01
B			
B	4.7679	14	GMOD02

Based on the findings in Table 13, Table 14 presents mean estimates for the different types of oil. Since the multiple comparison found oils GMOD01 and GMOD02 not to have significantly different means, a common mean value of New_WPD is estimated for these two oils.

Table 14: Mean Estimates for New_WPD

Oil	Mean
434-2	5.87214286
GMOD01 & GMOD02	5.01392857

Finally, Table 15 summarizes the mean estimates for log(PVIS), WPD, P_reten, and New_WPD by oil type. It also provides the sample standard deviation of each oil and variable individually.

Table 15: Mean Estimates with Standard Deviations

Variable	Oil	Mean	Standard Deviation
log(PVIS)	434-2	4.372330	0.330082
	GMOD01		0.175825
	GMOD02	4.060781	0.400314
WPD	434-2	6.818214	0.458706
	GMOD01		0.481160
	GMOD02	6.185714	0.550031
P_reten	434-2	73.809286	1.853821
	GMOD01	82.352143	3.122816
	GMOD02		3.480263
New_WPD	434-2	5.872143	0.607734
	GMOD01	5.013929	0.692898
	GMOD02		0.714522

4. ESTIMATNG THE STANDARD DEVIATION

This section proceeds by first estimating the variance components for log(PVIS), WPD, P_reten, and New_WPD and then using these estimates to calculate the standard deviation as well as reproducibility and repeatability for each variable. To estimate the variance components for log(PVIS), WPD, P_reten, and New_WPD, four separate ANOVA models are used, one for each variable. In contrast to the ANOVA models used in Section 3, the models in this section treat Oil, Lab, and Stand as random rather than fixed factors. This adjustment constitutes a change in the assumptions underlying the models. Whereas a fixed effects model is needed to perform multiple comparisons and estimate means, a random effects model, in which the variance of each factor is treated as a parameter, is necessary to estimate the variance components.

The same model is used to estimate the variance components for WPD, P_reten, and New_WPD. The preceding analysis showed that, in contrast to log(PVIS), the variable Stand does not have a significant effect on WPD, P_reten, or New_WPD, which means that Stand can be left out of the model used to estimate the variance components for these three variables. The model for WPD, P_reten, and New_WPD is as follows.

$$y_{ijk} = \mu + OIL_i + LAB_j + \varepsilon_{ijk}$$

$$i = 1,2,3$$

$$j = 1,2,3,4,5$$

k =1,2,3,4, for labs A and G

k = 1,2, for labs B, D, and F

$$\varepsilon_{ijk} \sim N(0, \sigma^2)\ i.i.d.$$

$$OIL_i \sim N(0, \sigma^2_{OIL})\ i.i.d.$$

$$LAB_j \sim N(0, \sigma^2_{LAB})\ i.i.d.$$

Table 16 presents the results from estimating the model with WPD as the response variable. The low *p*-values (< 0.05) for both Oil and Lab indicate that, as expected, the variance components associated with these two variables, σ^2_{OIL} and σ^2_{LAB}, are greater than zero.

Table 16: Random Effects ANOVA Results for WPD

The SAS System

The GLM Procedure

Source	Type III Expected Mean Square
Oil	Var(Error) + 14 Var(Oil)
Lab	Var(Error) + 8.1429 Var(Lab)

The SAS System

The GLM Procedure
Tests of Hypotheses for Random Model Analysis of Variance

Dependent Variable: WPD

Source	DF	Type III SS	Mean Square	F Value	Pr > F
Oil	2	4.616033	2.308017	12.44	<.0001
Lab	4	3.181887	0.795472	4.29	0.0063
Error: MS(Error)	35	6.496092	0.185603		

In order to find actual numerical values for the variance components of WPD, the first step is to note that taking the variance of both sides of the model listed above gives the following equation,

$$\sigma^2_{Total} = \sigma^2_{OIL} + \sigma^2_{LAB} + \sigma^2_{\varepsilon}.$$

The Type III Expected Mean Square column in Table 16 provides the formulas by which σ^2_{OIL}, σ^2_{LAB}, and σ^2_{ε} can be estimated. The formulas given by this table can be written as

$$E(MS_{OIL}) = \sigma^2_{\varepsilon} + 14\sigma^2_{OIL}$$

and

$$E(MS_{LAB}) = \sigma^2_{\varepsilon} + 8.1429\sigma^2_{LAB},$$

where $\sigma^2_{\varepsilon} = E(MSE)$.

The calculation of the coefficients for σ_{OIL}^2 and σ_{LAB}^2 in the equations for $E(MS_{OIL})$ and $E(MS_{LAB})$, respectively, are obtained using the equations for expected mean squares for unbalanced data.[1] The present data are unbalanced because not all types of observations have the same number of replicates (i.e., labs A and G have twice the number of replicates that labs B, D, and F have). The coefficient for σ_{OIL}^2 in the equation for $E(MS_{OIL})$ is thus calculated using the following expression,

$$\frac{N - k_1}{a - 1}$$

where *N* is the total number of observations, a is the number of oil types, and k_1 is given by the following formula in which $n_{i.}$ equals the number of observations for each oil type,

$$k_1 = \frac{1}{N}\sum_{i}^{a} n_{i.}^2 = \frac{1}{42}(14^2 + 14^2 + 14^2) = 14$$

Therefore,

$$\frac{N - k_1}{a - 1} = \frac{42 - 14}{2} = 14.$$

In the equation for $E(MS_{LAB})$, the coefficient for σ_{LAB}^2 is calculated from the following expression,

$$\frac{N - k_2}{b - 1}$$

where *b* is the number of labs, and k_2 is given by the following formula in which $n_{.j}$ equals the number of observations in each lab,

$$k_2 = \frac{1}{N}\sum_{j}^{b} n_{.j}^2 = \frac{1}{42}(12^2 + 12^2 + 6^2 + 6^2 + 6^2) = 9.4286$$

Therefore,

$$\frac{N - k_2}{b - 1} = \frac{42 - 9.4286}{4} = 8.1429.$$

Based on the mean square values in the SAS output in Table 16 and the above equations, point estimates of these variance components can be calculated as follows,

$$\hat{\sigma}_{\varepsilon}^2 = MSE = 0.185603,$$

$$\hat{\sigma}_{OIL}^2 = \frac{MS_{OIL} - MSE}{14} = \frac{2.308017 - 0.185603}{14} = 0.151601, \text{ and}$$

[1] These formulas are found in Hardeo Sahai and Mario Miguel Ojeda. 2005. *Analysis of Variance for Random Models, Volume II: Unbalanced Data*. Boston: Birkhauser, pp.167-169.

$$\hat{\sigma}^2_{LAB} = \frac{MS_{LAB} - MSE}{8.1429} = \frac{0.795472 - 0.185603}{8.1429} = 0.074896.$$

The total variance, σ^2_{Total}, is estimated as the sum of these different variance components

$$\hat{\sigma}^2_{Total} = \hat{\sigma}^2_{OIL} + \hat{\sigma}^2_{LAB} + \hat{\sigma}^2_{\varepsilon} = 0.412100.$$

These variance components are then used to calculate the standard deviation of Oil as well as reproducibility and repeatability for WPD. Repeatability is calculated based on the estimated variance of the error term, $\hat{\sigma}^2_{\varepsilon}$, while the calculation of reproducibility is based on the estimated variance of lab, $\hat{\sigma}^2_{LAB}$, and the number of degrees of freedom for Lab as reported in Table 16.

$$STD(Oil) = \sqrt{\hat{\sigma}^2_{OIL}} = \sqrt{0.151601} = 0.38936$$

$$Repeatability = 1.96 \times \sqrt{2} \times \sqrt{\hat{\sigma}^2_{\varepsilon}} = 1.96 \times \sqrt{2} \times \sqrt{0.185603} = 1.19416$$

$$Reproducibility = 1.96 \times \sqrt{df_{LAB}} \times \sqrt{\hat{\sigma}^2_{LAB}} = 1.96 \times \sqrt{4} \times \sqrt{0.074896} = 1.07279$$

Table 17 presents the results obtained from estimating the model with P_reten as the response variable. The low *p*-values (< 0.05) for both Oil and Lab indicate that the variance components for these two variables, σ^2_{OIL} and σ^2_{LAB}, are greater than zero.

Table 17: Random Effects ANOVA Results for P_reten

The SAS System

The GLM Procedure

Source	Type III Expected Mean Square
Oil	Var(Error) + 14 Var(Oil)
Lab	Var(Error) + 8.1429 Var(Lab)

The SAS System

The GLM Procedure
Tests of Hypotheses for Random Model Analysis of Variance

Dependent Variable: P_reten

Source	DF	Type III SS	Mean Square	F Value	Pr > F
Oil	2	688.050833	344.025417	65.84	<.0001
Lab	4	146.025890	36.506473	6.99	0.0003
Error: MS(Error)	35	182.885317	5.225295		

As with the variance components of WPD, the variance components of P_reten are related to one another by the following equation,

$$\sigma^2_{Total} = \sigma^2_{OIL} + \sigma^2_{LAB} + \sigma^2_{\varepsilon}.$$

The Type III Expected Mean Square column given by SAS in Table 17 provides the formulas by which the values of σ^2_{OIL}, σ^2_{LAB} and σ^2_{ε} are estimated. Specifically,

$$\hat{\sigma}^2_{\varepsilon} = MSE = 5.225295,$$

$$\hat{\sigma}^2_{OIL} = \frac{MS_{OIL} - MSE}{14} = \frac{344.025417 - 5.225295}{14} = 24.20001, \text{ and}$$

$$\hat{\sigma}^2_{LAB} = \frac{MS_{LAB} - MSE}{8.1429} = \frac{36.506473 - 5.225295}{8.1429} = 3.841528.$$

The total variance, σ^2_{Total}, can be estimated as,

$$\hat{\sigma}^2_{Total} = \hat{\sigma}^2_{OIL} + \hat{\sigma}^2_{LAB} + \hat{\sigma}^2_{\varepsilon} = 33.26683.$$

These variance components are then used to calculate the standard deviation of Oil as well as the reproducibility and repeatability for P_reten. Similarly to WPD, repeatability is calculated based on the estimated variance of the error term, $\hat{\sigma}^2_{\varepsilon}$, while the calculation of reproducibility is based on the estimated variance of lab, $\hat{\sigma}^2_{LAB}$, and the number of degrees of freedom for Lab as reported in Table 17.

$$STD(Oil) = \sqrt{\hat{\sigma}^2_{OIL}} = \sqrt{24.20001} = 4.91935$$

$$Repeatability = 1.96 \times \sqrt{2} \times \sqrt{5.225295} = 6.33616$$

$$Reproducibility = 1.96 \times \sqrt{4} \times \sqrt{3.841528} = 7.68313$$

Table 18 presents the results obtained from estimating the model with New_WPD as the response variable. The low p-values (< 0.05) for both Oil and Lab indicate that the variance components for these two variables, σ^2_{OIL} and σ^2_{LAB}, are greater than zero.

Table 18: Random Effects ANOVA Results for New_WPD

The SAS System

The GLM Procedure

Source	Type III Expected Mean Square
Oil	Var(Error) + 14 Var(Oil)
Lab	Var(Error) + 8.1429 Var(Lab)

The SAS System

The GLM Procedure
Tests of Hypotheses for Random Model Analysis of Variance

Dependent Variable: New_WPD

Source	DF	Type III SS	Mean Square	F Value	Pr > F
Oil	2	8.569729	4.284864	12.78	<.0001
Lab	4	5.942100	1.485525	4.43	0.0053
Error: MS(Error)	35	11.737771	0.335365		

As with the variance components of WPD and P_reten, the variance components of New_WPD are related to one another by the following equation,

$$\sigma_{Total}^2 = \sigma_{OIL}^2 + \sigma_{LAB}^2 + \sigma_{\varepsilon}^2.$$

The Type III Expected Mean Square column given by SAS in Table 18 provides the formulas by which the values of σ_{OIL}^2, σ_{LAB}^2 and σ_{ε}^2 are estimated. Specifically,

$$\hat{\sigma}_{\varepsilon}^2 = MSE = 0.335365,$$

$$\hat{\sigma}_{OIL}^2 = \frac{MS_{OIL} - MSE}{14} = \frac{4.284864 - 0.335365}{14} = 0.282107, \text{ and}$$

$$\hat{\sigma}_{LAB}^2 = \frac{MS_{LAB} - MSE}{8.1429} = \frac{1.485525 - 0.335365}{8.1429} = 0.141247.$$

The total variance, σ_{Total}^2, can be estimated as

$$\hat{\sigma}_{Total}^2 = \hat{\sigma}_{OIL}^2 + \hat{\sigma}_{LAB}^2 + \hat{\sigma}_{\varepsilon}^2 = 0.758719.$$

These variance components are then used to calculate the standard deviation of Oil as well as the reproducibility and repeatability for New_WPD. Similarly to WPD and P_reten, repeatability is calculated based on the estimated variance of the error term, $\hat{\sigma}_{\varepsilon}^2$, while the calculation of reproducibility is based on the estimated variance of lab, $\hat{\sigma}_{LAB}^2$, and the number of degrees of freedom for Lab as reported in Table 18.

$$STD(Oil) = \sqrt{\hat{\sigma}_{OIL}^2} = \sqrt{0.282107} = 0.53114$$

$$Repeatability = 1.96 \times \sqrt{2} \times \sqrt{0.335365} = 1.60520$$

$$Reproducibility = 1.96 \times \sqrt{4} \times \sqrt{0.141247} = 1.47325$$

The estimation of the variance components of log(PVIS) requires a different model from that used to estimate the variance components of WPD, P_reten, and New_WPD. Since the analysis in the preceding section found that Stand had a significant effect on log(PVIS), Stand must be included in the model for log(PVIS). In this model, Oil, Lab, and Stand are all random effects, and Stand is nested in Lab. As in the fixed effects models in Section 3, Stand is treated as nested in Lab because the stands are specific to the labs in which they are located and, thus, cannot be treated as identical across labs. The model for log(PVIS) can be written as,

$$y_{ijk} = \mu + OIL_i + LAB_j + STAND_{k(j)} + \varepsilon_{(ijk)l}$$

$$i = 1,2,3$$

$$j = 1,2,3,4,5$$

k =1,2, for labs A and G

k =1, for labs B, D, and F

l =1,2

$$\varepsilon_{(ijk)l} \sim N(0, \sigma^2)\ i.i.d.$$

$$OIL_i \sim N(0, \sigma_{OIL}^2)\ i.i.d.$$

$$LAB_j \sim N(0, \sigma_{LAB}^2)\ i.i.d.$$

$$STAND_{k(j)} \sim N\left(0, \sigma_{STAND(LAB)}^2\right)\ i.i.d.$$

Table 19 presents the results obtained from estimating this model. The low *p*-values (< 0.05) for Oil and Stand indicate that the variance components for these two variables, σ_{OIL}^2 and $\sigma_{STAND(LAB)}^2$, are greater than zero. The high *p*-value for Lab, on the other hand, indicates that the variance component for Lab, σ_{LAB}^2, is not significantly different from zero.

Table 19: Random Effects ANOVA Results for log(PVIS)

The SAS System

The GLM Procedure

Source	Type III Expected Mean Square
Oil	Var(Error) + 14 Var(Oil)
Lab	Var(Error) + 6 Var(Stand(Lab)) + 8.1429 Var(Lab)
Stand(Lab)	Var(Error) + 6 Var(Stand(Lab))

The SAS System

The GLM Procedure
Tests of Hypotheses for Random Model Analysis of Variance

Dependent Variable: LogPVIS

Source	DF	Type III SS	Mean Square	F Value	Pr > F
Oil	2	0.982248	0.491124	8.23	0.0013
Stand(Lab)	2	0.592637	0.296319	4.96	0.0130
Error: MS(Error)	33	1.969666	0.059687		

Source	DF	Type III SS	Mean Square	F Value	Pr > F
Lab	4	1.339253	0.334813	1.13	0.5194
Error	2	0.592637	0.296319		
Error: MS(Stand(Lab))					

The variance components of log(PVIS) are related to one another by the following equation, which is found by taking the variance of both sides of the model listed earlier for log(PVIS),

$$\sigma^2_{Total} = \sigma^2_{OIL} + \sigma^2_{LAB} + \sigma^2_{STAND(LAB)} + \sigma^2_{\varepsilon}.$$

The Type III Expected Mean Square column in Table 19 provides the formulas by which σ^2_{OIL}, σ^2_{LAB}, $\sigma^2_{STAND(LAB)}$, and σ^2_{ε} are estimated.

$$\hat{\sigma}^2_{\varepsilon} = MSE = 0.059687,$$

$$\hat{\sigma}^2_{OIL} = \frac{MS_{OIL} - MSE}{14} = \frac{0.491124 - 0.059687}{14} = 0.03081693,$$

$$\hat{\sigma}^2_{STAND(LAB)} = \frac{MS_{STAND(LAB)} - MSE}{6} = \frac{0.296319 - 0.059687}{6} = 0.03943867, \text{ and}$$

$$\hat{\sigma}^2_{LAB} = \frac{MS_{LAB} - MS_{STAND(LAB)}}{8.1429} = \frac{0.334813 - 0.296319}{8.1429} = 0.004727308.$$

The equation for estimating σ^2_{LAB} can be obtained by noting that since

$$E\left(MS_{STAND(LAB)}\right) = \sigma^2_{\varepsilon} + 6\sigma^2_{STAND(LAB)},$$

the equation for the expected mean square of lab, which is

$$E(MS_{LAB}) = \sigma_\varepsilon^2 + 6\sigma_{STAND(LAB)}^2 + 8.1429\sigma_{LAB}^2,$$

can be simplified to the following

$$E(MS_{LAB}) = E\left(MS_{STAND(LAB)}\right) + 8.1429\sigma_{LAB}^2,.$$

Based on the above calculations, the total variance, σ_{Total}^2, can then be estimated as:

$$\hat{\sigma}_{Total}^2 = \hat{\sigma}_{OIL}^2 + \hat{\sigma}_{LAB}^2 + \hat{\sigma}_{STAND(LAB)}^2 + \hat{\sigma}_\varepsilon^2 = 0.1346699.$$

These variance components are then used to calculate the standard deviation of Oil as well as the reproducibility and repeatability for log(PVIS). Similarly to WPD, P_reten, and New_WPD, repeatability is calculated based on the estimated variance of the error term, $\hat{\sigma}_\varepsilon^2$. However, given the different model used to calculate the variance components of log(PVIS), the calculation of reproducibility for log(PVIS) differs from that used for the previous three variables, as it takes into account the estimated variances of both Lab, $\hat{\sigma}_{LAB}^2$, and Stand, $\hat{\sigma}_{STAND(LAB)}^2$.

$$STD(Oil) = \sqrt{\hat{\sigma}_{OIL}^2} = \sqrt{0.03081693} = 0.17555$$

$$Repeatability = 1.96 \times \sqrt{2} \times \sqrt{0.059687} = 0.67719$$

$$Reproducibility = 1.96 \times \sqrt{4} \times \sqrt{\hat{\sigma}_{LAB}^2 + \hat{\sigma}_{STAND(LAB)}^2} = 1.96 \times \sqrt{4} \times \sqrt{0.004727308 + 0.03943867}$$

$$= 0.82382$$

Finally, Tables 20 and 21 summarize the findings from this section. Table 20 reports the variance components of log(PVIS), WPD, P_reten, and New_WPD. Table 21 presents the standard deviation of Oil as well as reproducibility and repeatability for each variable.

Table 20: Variance Component Estimates

	log(PVIS)	WPD	P_reten	New_WPD
Var(Oil)	0.030817	0.151601	24.20001	0.282107
Var(Lab)	0.004727	0.074896	3.841528	0.141247
Var(Stand)	0.039439	--	--	--
Var(Error)	0.059687	0.185603	5.225295	0.335365
Total variance	0.134670	0.412100	33.26683	0.758719

Table 21: Standard Deviation, Reproducibility, and Repeatability

	log(PVIS)	WPD	P_reten	New_WPD
STD(Oil)	0.17555	0.38936	4.91935	0.53114
Reproducibility	0.82382	1.07279	7.68313	1.47325
Repeatability	0.67719	1.19416	6.33616	1.60520

5. CONCLUSION

This report has analyzed the oil test data provided by General Motors. It began with an exploratory analysis of the distribution of the four variables, PVIS, WPD, P_reten, and New_WPD. The main result of this section was that PVIS was transformed into its natural logarithm, log(PVIS), in order to correct its non-normal distribution. An outlier was also discovered, but subsequent analysis determined that this outlier did not have a significant effect on the results of the analysis.

Estimation of the means of log(PVIS), WPD, P_reten, and New_WPD by oil type was carried out with four fixed effects ANOVA models, one for each variable. All four models, which included Oil, Lab, and Stand as factors, were found to be statistically significant. Oil and Lab were also found to be significant in all four models, although Stand was only significant in the model for log(PVIS). Since Oil was significant in all four models – indicating that the mean of each variable differs by oil type – four sets of multiple comparisons were undertaken to determine exactly how the mean value of each variable differed by Oil. These multiple comparisons demonstrated that oil GMOD02 has a lower mean log(PVIS) and lower mean WPD than both of the other two oils and that oil 434-2 has a significantly lower mean P_reten value and a significantly higher New_WPD value than do the other two oils. These results then allowed estimation of the mean of each variable by oil type.

The estimation of the standard deviations of the four variables, taking into account reproducibility (lab-to-lab variability) and repeatability (variability between replicates), was carried out with four random effects ANOVA models, one for each variable. Since Stand was found not to have a significant effect on WPD, P_reten, or New_WPD, the variance components estimated for these two variables were limited to variance by Lab, variance by Oil, and random error. In contrast, since Stand did have a significant effect on log(PVIS), variance by Stand was estimated for this variable in addition to variance by Lab, variance by Oil, and random error. The variance components estimated for each variable were then used to calculate its standard deviation as well as reproducibility and repeatability.

APPENDIX I: MULTIPLE COMPARISONS BY LAB

Table 22 presents results for multiple comparisons of mean log(PVIS) value by Lab. The results indicate that lab B is different from labs G and F in regard to log(PVIS), while lab F is also different from labs A and D.

Table 22: Comparison of log(PVIS) Means by Lab

Tukey's Studentized Range (HSD) Test for LogPVIS

Note: This test controls the Type I experimentwise error rate.

Alpha	0.05
Error Degrees of Freedom	33
Error Mean Square	0.059687
Critical Value of Studentized Range	4.07897

Comparisons significant at the 0.05 level are indicated by ***.

Lab Comparison	Difference Between Means	Simultaneous 95% Confidence Limits		
B - D	0.17667	-0.23016	0.58351	
B - A	0.23432	-0.11800	0.58665	
B - G	0.36210	0.00977	0.71442	***
B - F	0.62149	0.21466	1.02832	***
D - B	-0.17667	-0.58351	0.23016	
D - A	0.05765	-0.29468	0.40997	
D - G	0.18542	-0.16690	0.53775	
D - F	0.44482	0.03799	0.85165	***
A - B	-0.23432	-0.58665	0.11800	
A - D	-0.05765	-0.40997	0.29468	
A - G	0.12778	-0.15990	0.41545	
A - F	0.38717	0.03484	0.73950	***
G - B	-0.36210	-0.71442	-0.00977	***
G - D	-0.18542	-0.53775	0.16690	
G - A	-0.12778	-0.41545	0.15990	
G - F	0.25939	-0.09293	0.61172	
F - B	-0.62149	-1.02832	-0.21466	***
F - D	-0.44482	-0.85165	-0.03799	***
F - A	-0.38717	-0.73950	-0.03484	***
F - G	-0.25939	-0.61172	0.09293	

Table 23 presents results for multiple comparisons of mean WPD value by Lab. The results indicate that lab D is significantly different from labs B and G in regard to mean WPD value.

Table 23: Comparison of WPD Means by Lab

Tukey's Studentized Range (HSD) Test for WPD

Note: This test controls the Type I experimentwise error rate.

Alpha	0.05
Error Degrees of Freedom	33
Error Mean Square	0.195523
Critical Value of Studentized Range	4.07897

Comparisons significant at the 0.05 level are indicated by ***.

Lab Comparison	Difference Between Means	Simultaneous 95% Confidence Limits		
D - F	0.2567	-0.4797	0.9930	
D - A	0.6017	-0.0360	1.2393	
D - G	0.7542	0.1165	1.3918	***
D - B	0.7833	0.0470	1.5197	***
F - D	-0.2567	-0.9930	0.4797	
F - A	0.3450	-0.2927	0.9827	
F - G	0.4975	-0.1402	1.1352	
F - B	0.5267	-0.2097	1.2630	
A - D	-0.6017	-1.2393	0.0360	
A - F	-0.3450	-0.9827	0.2927	
A - G	0.1525	-0.3682	0.6732	
A - B	0.1817	-0.4560	0.8193	
G - D	-0.7542	-1.3918	-0.1165	***
G - F	-0.4975	-1.1352	0.1402	
G - A	-0.1525	-0.6732	0.3682	
G - B	0.0292	-0.6085	0.6668	
B - D	-0.7833	-1.5197	-0.0470	***
B - F	-0.5267	-1.2630	0.2097	
B - A	-0.1817	-0.8193	0.4560	
B - G	-0.0292	-0.6668	0.6085	

Table 24 presents results for multiple comparisons of mean P_reten value by lab. The results indicate that lab A is significantly different from labs F and G in regard to mean P_reten value.

Table 24: Comparison of P_reten Means by Lab

Tukey's Studentized Range (HSD) Test for P_reten

Note: This test controls the Type I experimentwise error rate.

Alpha	0.05
Error Degrees of Freedom	33
Error Mean Square	5.352395
Critical Value of Studentized Range	4.07897

Comparisons significant at the 0.05 level are indicated by ***.

Lab Comparison	Difference Between Means	Simultaneous 95% Confidence Limits		
F - G	1.0267	-2.3097	4.3631	
F - B	2.1200	-1.7326	5.9726	
F - D	2.2433	-1.6092	6.0959	
F - A	5.1117	1.7753	8.4481	***
G - F	-1.0267	-4.3631	2.3097	
G - B	1.0933	-2.2431	4.4297	
G - D	1.2167	-2.1197	4.5531	
G - A	4.0850	1.3608	6.8092	***
B - F	-2.1200	-5.9726	1.7326	
B - G	-1.0933	-4.4297	2.2431	
B - D	0.1233	-3.7292	3.9759	
B - A	2.9917	-0.3447	6.3281	
D - F	-2.2433	-6.0959	1.6092	
D - G	-1.2167	-4.5531	2.1197	
D - B	-0.1233	-3.9759	3.7292	
D - A	2.8683	-0.4681	6.2047	
A - F	-5.1117	-8.4481	-1.7753	***
A - G	-4.0850	-6.8092	-1.3608	***
A - B	-2.9917	-6.3281	0.3447	
A - D	-2.8683	-6.2047	0.4681	

Table 25 presents results for multiple comparisons of mean New_WPD value by lab. The results indicate that lab D is significantly different from labs A and G in regard to mean New_WPD value.

Table 25: Comparison of New_WPD Means by Lab

Tukey's Studentized Range (HSD) Test for New_WPD

Note: This test controls the Type I experimentwise error rate.

Alpha	0.05
Error Degrees of Freedom	33
Error Mean Square	0.353624
Critical Value of Studentized Range	4.07897

Comparisons significant at the 0.05 level are indicated by ***.

Lab Comparison	Difference Between Means	Simultaneous 95% Confidence Limits		
D - F	0.4517	-0.5386	1.4419	
D - A	0.9567	0.0991	1.8143	***
D - B	0.9583	-0.0319	1.9486	
D - G	1.0800	0.2224	1.9376	***
F - D	-0.4517	-1.4419	0.5386	
F - A	0.5050	-0.3526	1.3626	
F - B	0.5067	-0.4836	1.4969	
F - G	0.6283	-0.2293	1.4859	
A - D	-0.9567	-1.8143	-0.0991	***
A - F	-0.5050	-1.3626	0.3526	
A - B	0.0017	-0.8559	0.8593	
A - G	0.1233	-0.5769	0.8235	
B - D	-0.9583	-1.9486	0.0319	
B - F	-0.5067	-1.4969	0.4836	
B - A	-0.0017	-0.8593	0.8559	
B - G	0.1217	-0.7359	0.9793	
G - D	-1.0800	-1.9376	-0.2224	***
G - F	-0.6283	-1.4859	0.2293	
G - A	-0.1233	-0.8235	0.5769	
G - B	-0.1217	-0.9793	0.7359	

APPENDIX II: CHECKING VALIDITY OF ANOVA ASSUMPTIONS

This appendix uses a series of residual plots to assess the validity of the assumptions underlying the four ANOVA models used in Section 3. The model with log(PVIS) as the response is examined first. Figure 15 presents the normal probability plot of this model's residuals, and this plot indicates slight departure from normality. However, this departure is small enough that the Anderson-Darling test (p-value = 0.088) does not reject the hypothesis if normality at $\alpha = 0.05$. It is noteworthy that, to the extent that the plot below departs from normality, this departure is due primarily to the observation in the upper-right corner, which is the outlier, observation 41.

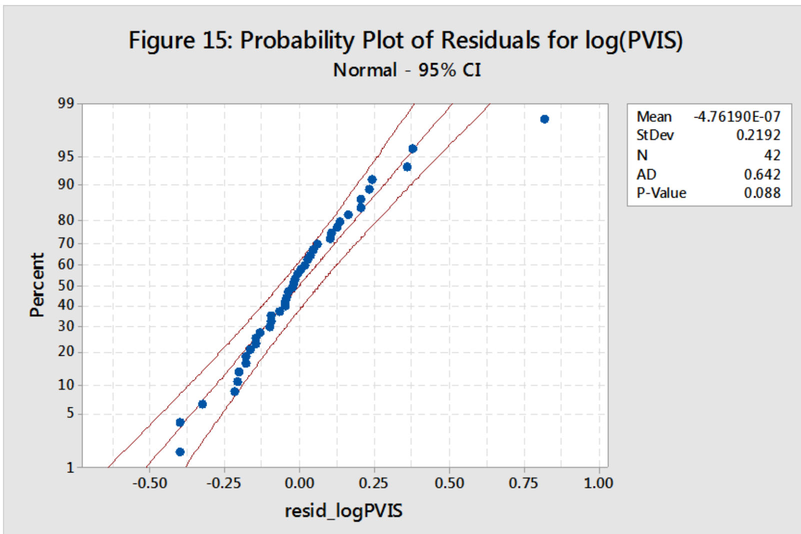


Figure 16 presents a plot of the model's residuals against the predicted values of log(PVIS), while Figures 17, 18, 19, and 20 present plot these residuals against the variables Oil, Lab, and Stand. Since Stand is nested in Lab, there are two plots for this variable – one for each of the labs with multiple stands, which are labs A and G. None of the plots in any of these figures reveals anything problematic. The residuals appear to have equal variance for different values of the grouping variables and for different values of the predicted outcome, indicating that the assumption of constant variance of the error term is valid.

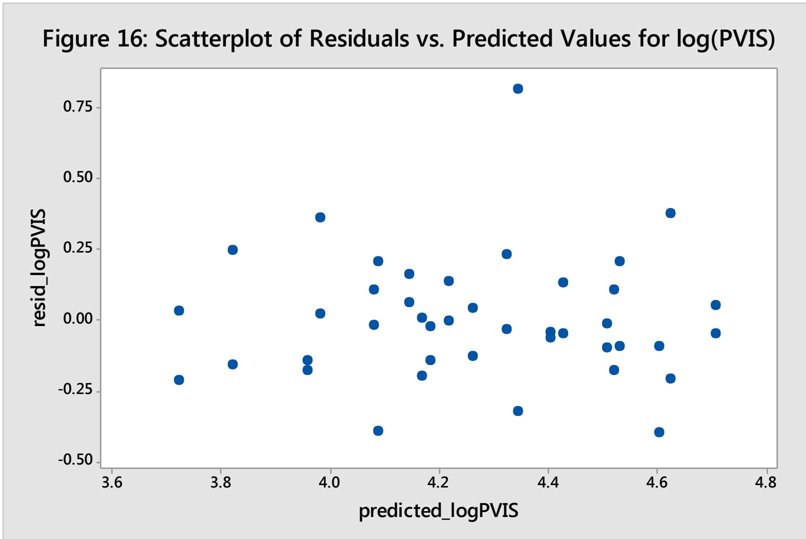


Figure 16: Scatterplot of Residuals vs. Predicted Values for log(PVIS)
resid_logPVIS
0.75
0.50
0.25
0.00
-0.25
-0.50
3.6
3.8
4.0
4.2
4.4
4.6
4.8
predicted_logPVIS

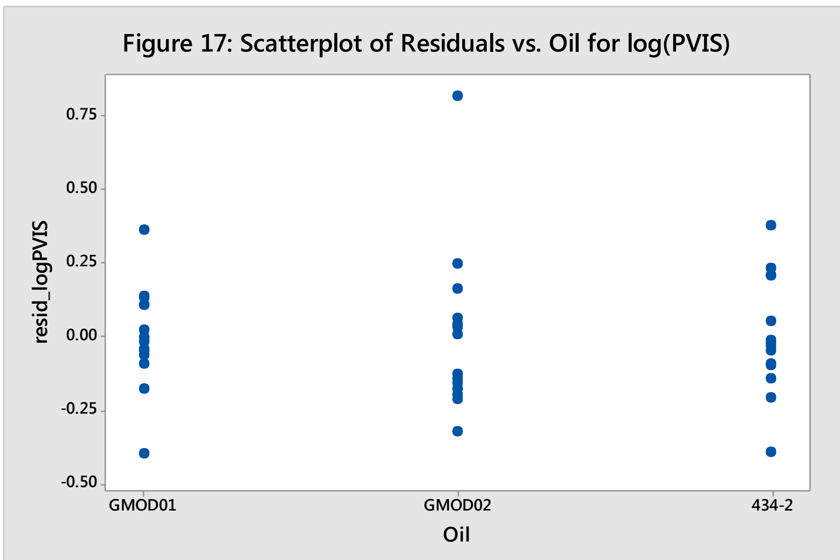


Figure 17: Scatterplot of Residuals vs. Oil for log(PVIS)
resid_logPVIS
0.75
0.50
0.25
0.00
-0.25
-0.50
GMOD01
GMOD02
434-2
Oil

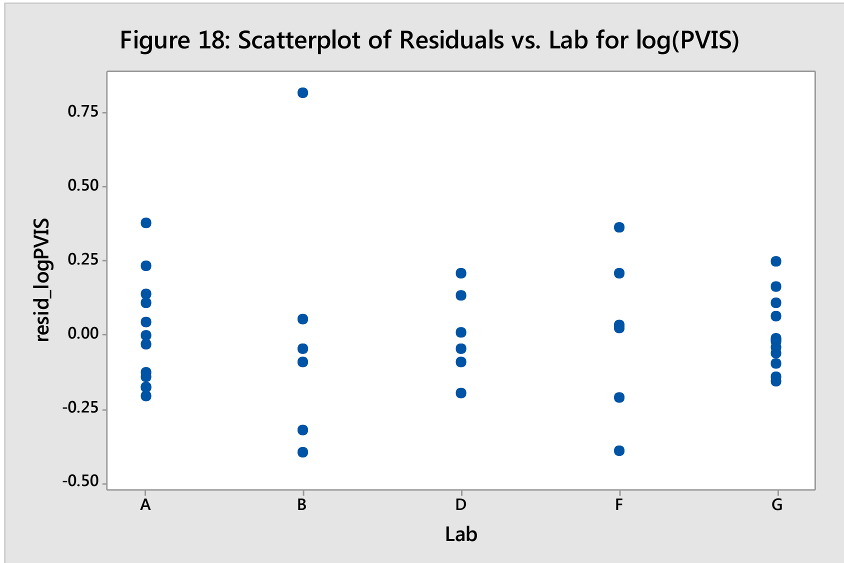


Figure 18: Scatterplot of Residuals vs. Lab for log(PVIS)
0.75
0.50
0.25
0.00
-0.25
-0.50
resid_logPVIS
A
B
D
F
G
Lab

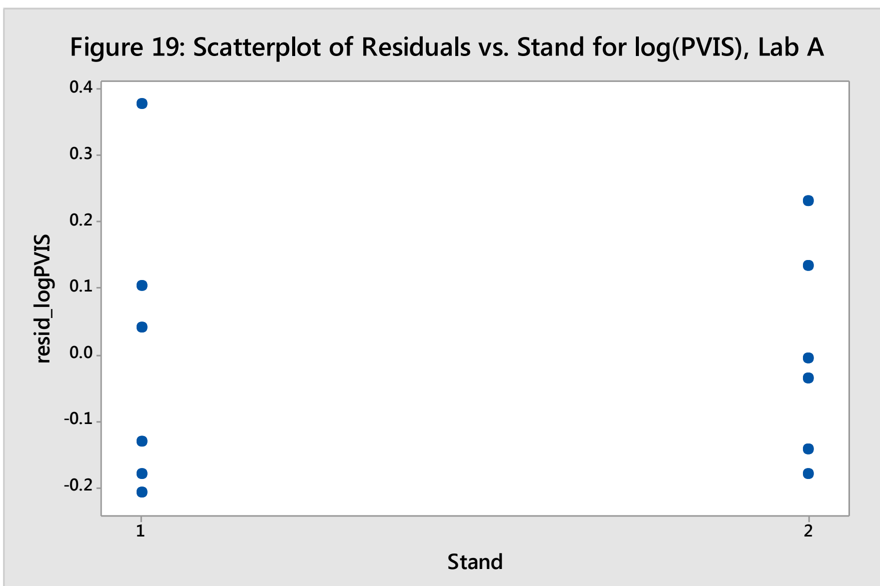


Figure 19: Scatterplot of Residuals vs. Stand for log(PVIS), Lab A
0.4
0.3
0.2
0.1
0.0
-0.1
-0.2
resid_logPVIS
1
2
Stand

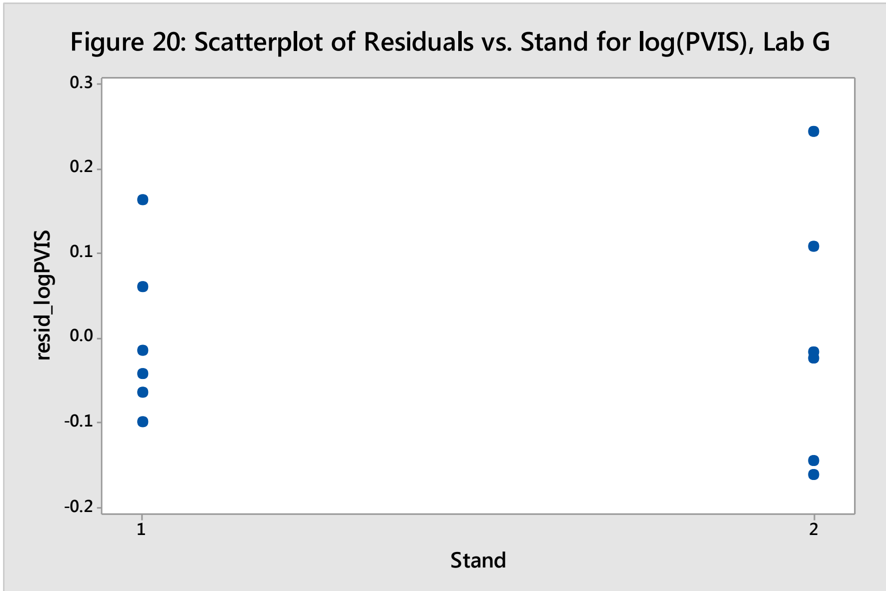
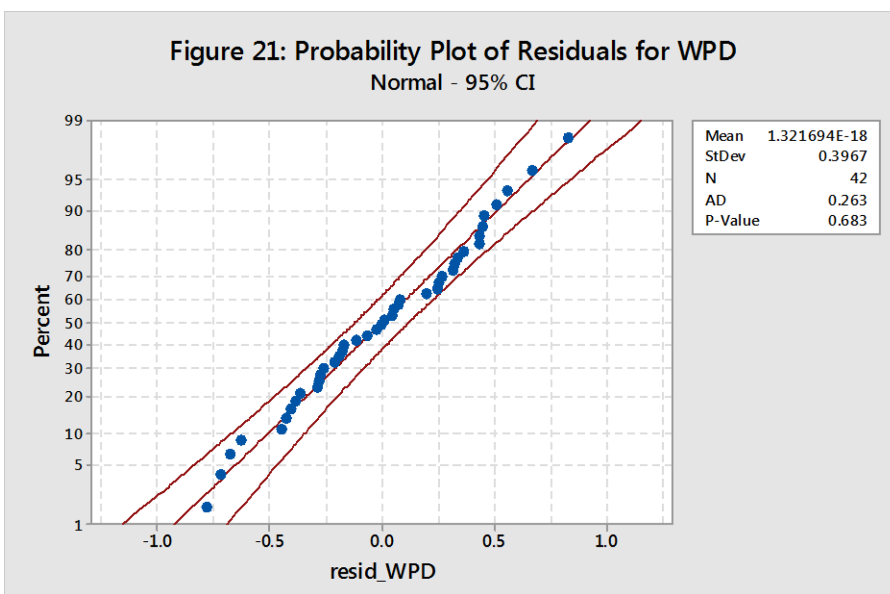


Figure 21 presents the normal probability plot of the residuals from the model with WPD as response, and it shows no major departures from normality, indicating that the assumption of a normally distributed error term is valid.



The plots of the residuals against Oil, Lab, and Stand and against the predicted values of WPD, presented in Figures 22, 23, 24, 25, and 26, reveal nothing particularly troublesome. The residuals based on observations from labs D and F appear to have slightly lower variance than the residuals based on the observations from the other three labs. Nonetheless, there do not appear to be major departures from the assumption of an error term with constant variance.

Figure 22: Scatterplot of Residuals vs. Predicted Values for WPD

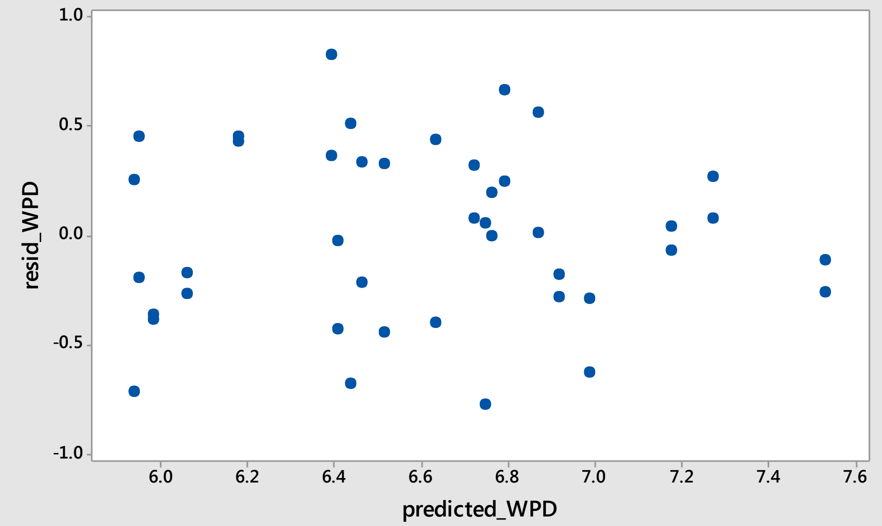
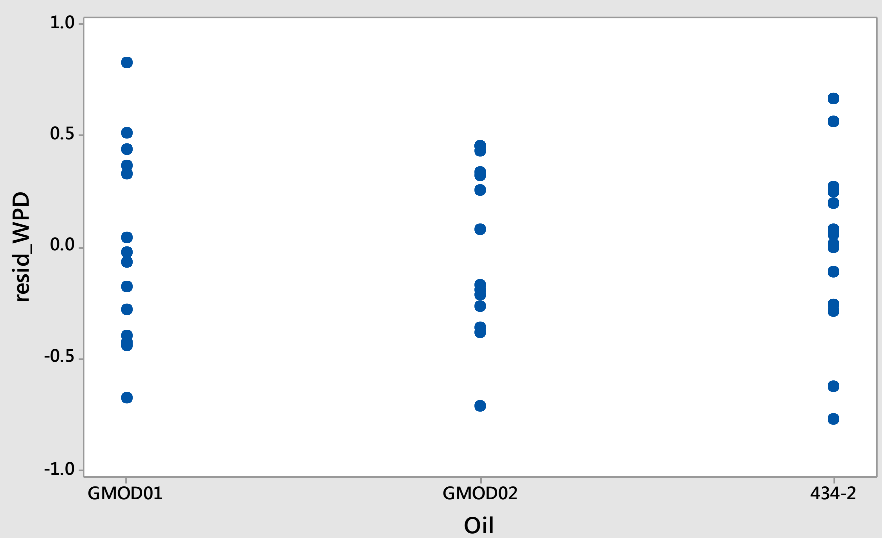


Figure 23: Scatterplot of Residuals vs. Oil for WPD



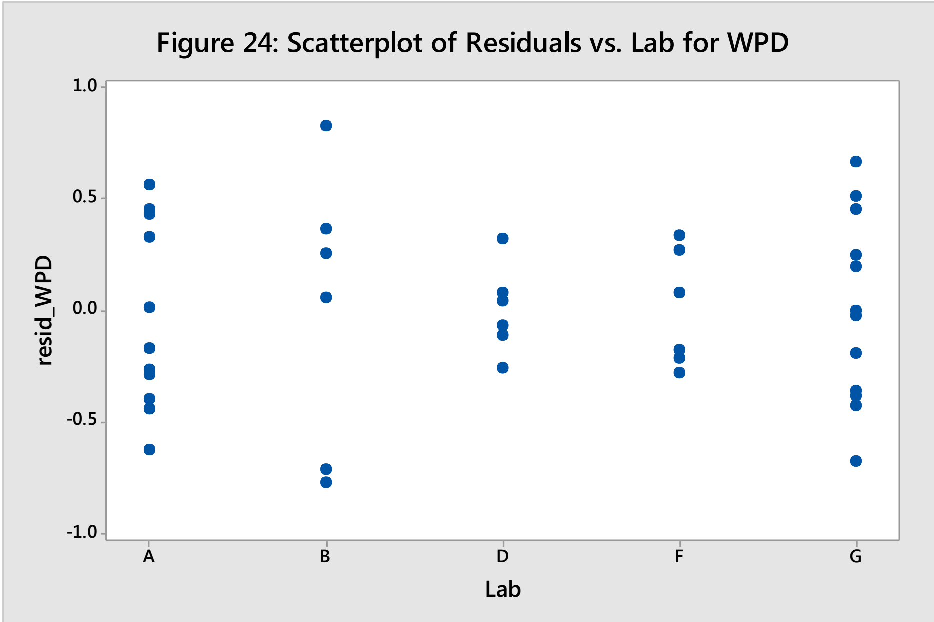


Figure 24: Scatterplot of Residuals vs. Lab for WPD
1.0
0.5
0.0
-0.5
-1.0
resid_WPD
A
B
D
F
G
Lab

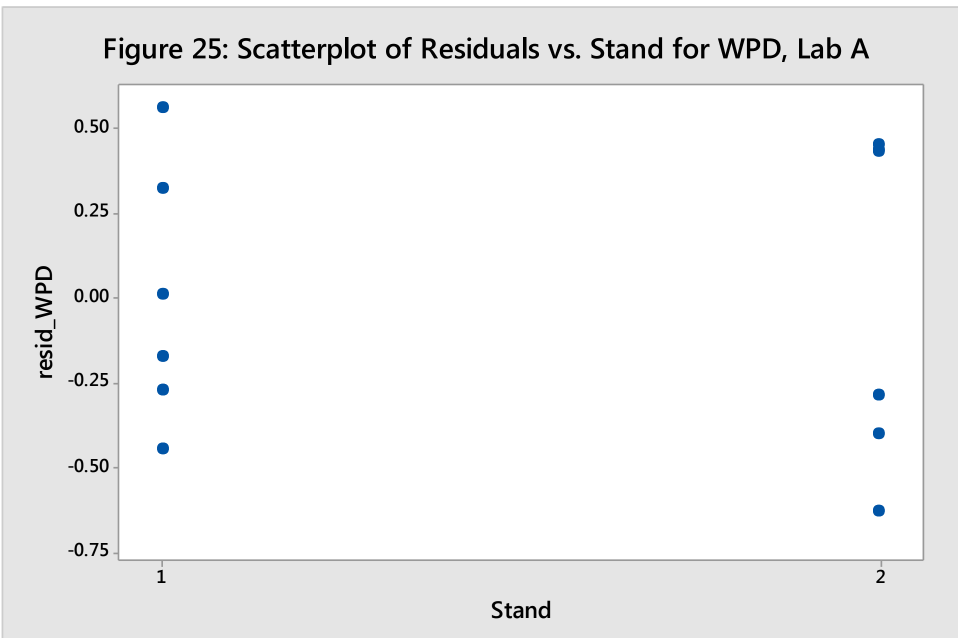


Figure 25: Scatterplot of Residuals vs. Stand for WPD, Lab A
0.50
0.25
0.00
-0.25
-0.50
-0.75
resid_WPD
1
2
Stand

Figure 26: Scatterplot of Residuals vs. Stand for WPD, Lab G

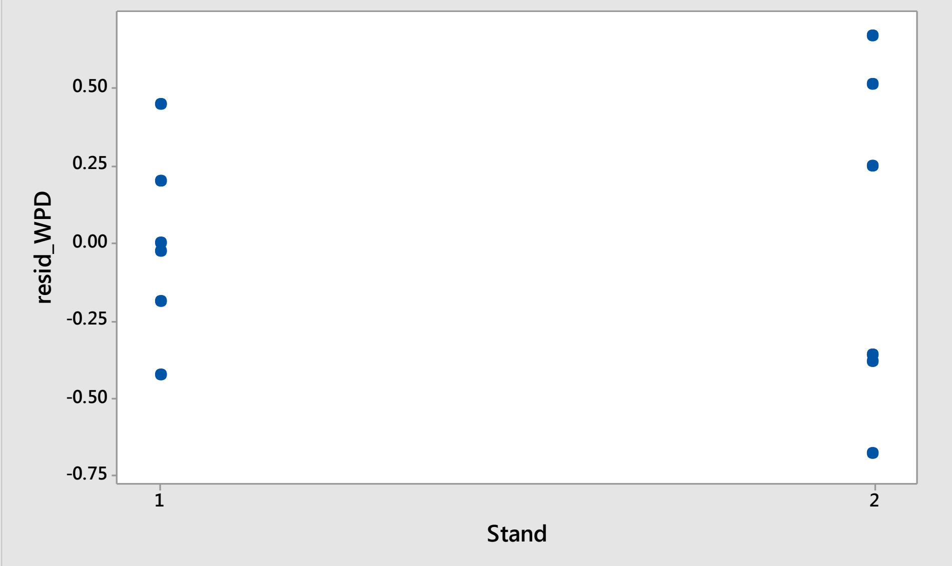
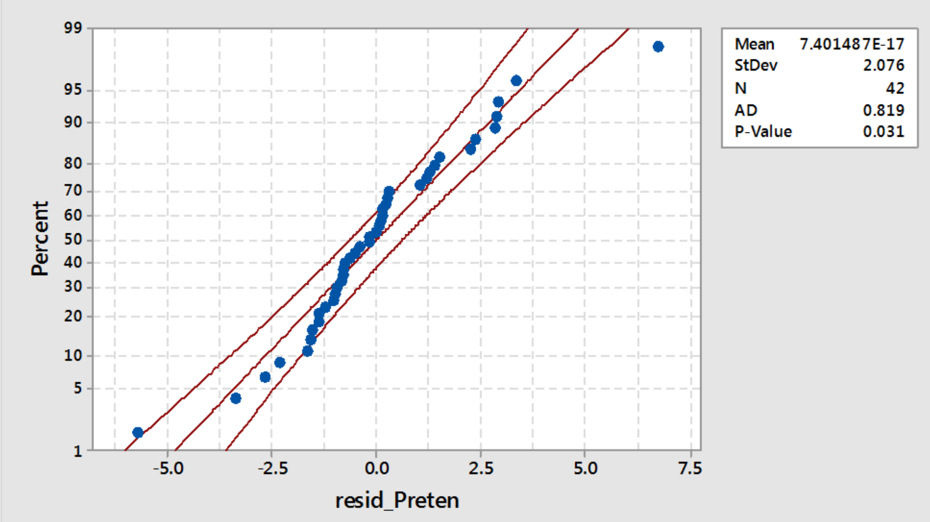


Figure 27 presents the normal probability plot of the residuals from the model with P_reten as the response. It indicates some departure from normality, as extreme observations at either end of the plot create something of an S-shape. The Anderson-Darling test rejects the hypothesis of normality with a p-value of 0.031. It is noteable that the plot would look significantly more normal without the observation in the upper-right corner, which is the outlier, observation 41.

Figure 27: Probability Plot of Residuals for P_reten

Normal - 95% CI



The plots of the residuals against Oil, Lab, and Stand and against the predicted response, presented in Figures 28, 29, 30, 31, and 32, have one potentially problematic feature. The plot of the residuals against lab indicates that labs B and D have somewhat different variances, as the residuals for lab B have a wide spread and those for lab D are closely packed together. However, these findings are unlikely to create major problems for the analysis.

Figure 28: Scatterplot of Residuals vs. Predicted Values for P_reten

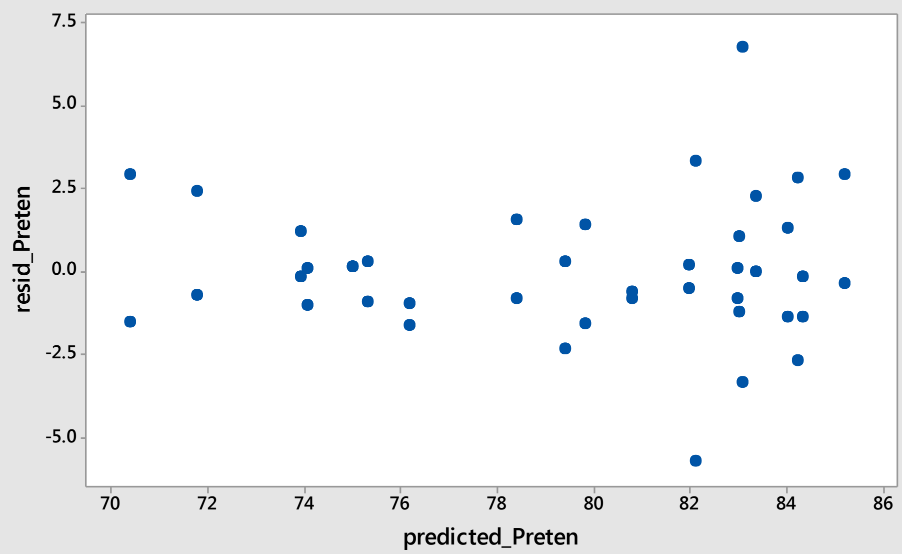
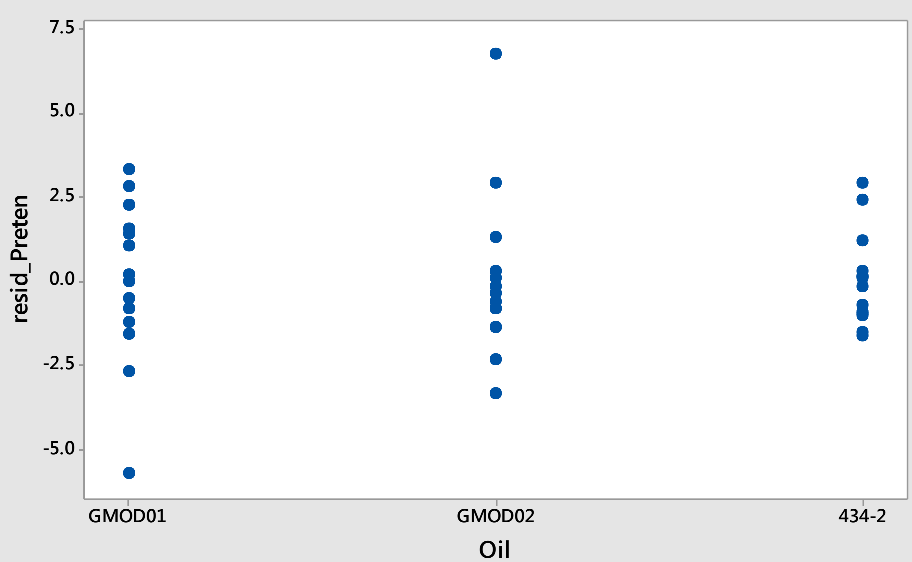


Figure 29: Scatterplot of Residuals vs. Oil for P_reten



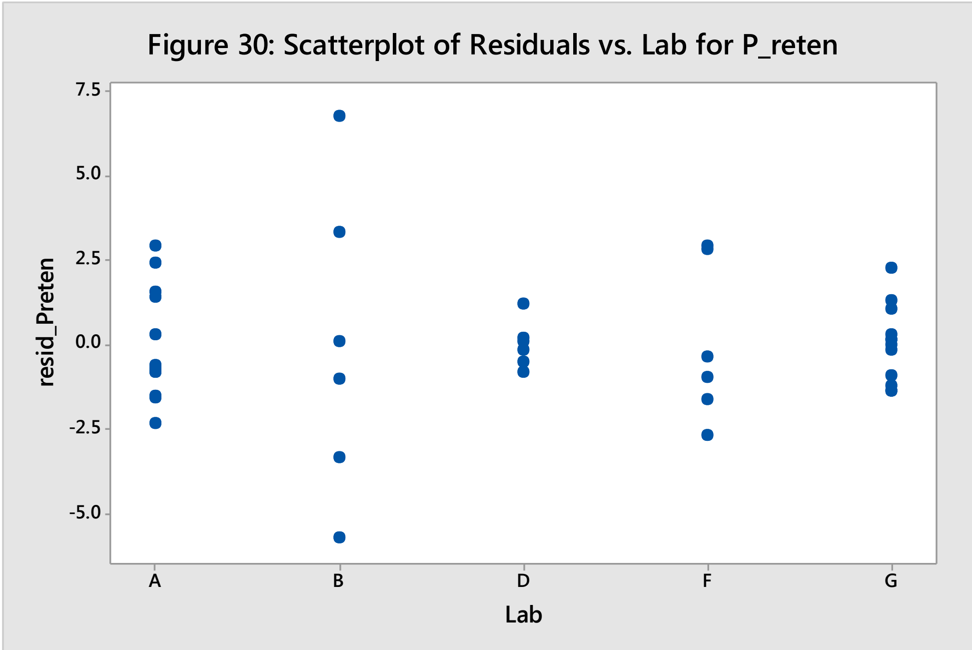


Figure 30: Scatterplot of Residuals vs. Lab for P_reten
resid_Preten
7.5
5.0
2.5
0.0
-2.5
-5.0
A
B
D
F
G
Lab

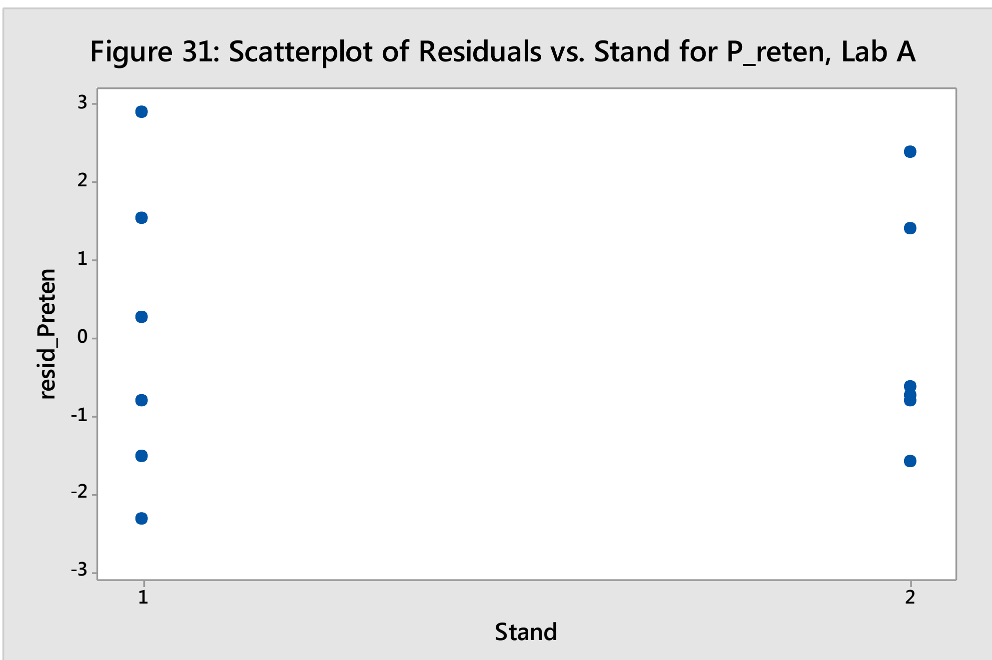


Figure 31: Scatterplot of Residuals vs. Stand for P_reten, Lab A
resid_Preten
3
2
1
0
-1
-2
-3
1
2
Stand

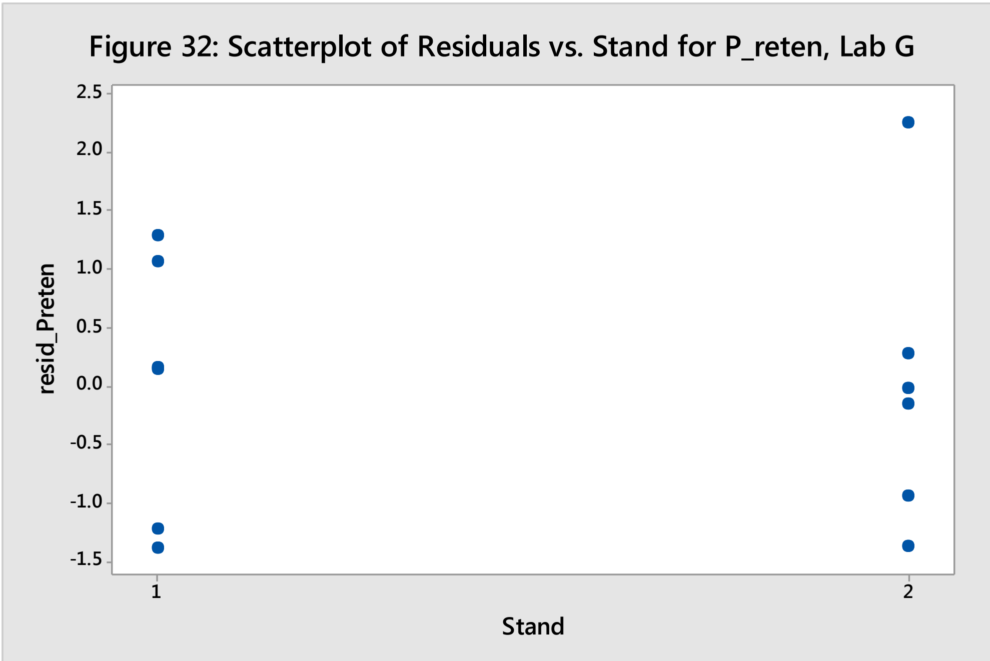
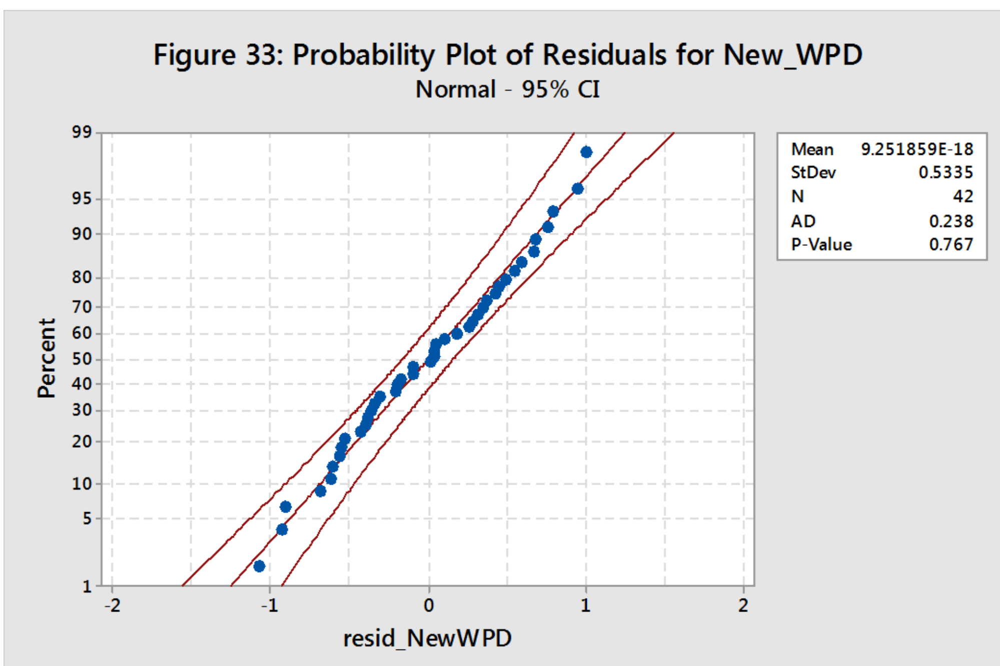


Figure 33 presents the normal probability plot of the residuals from the model with New_WPD as response, and it shows no major departures from normality, indicating that the assumption of a normally distributed error term is valid.



The plots of the residuals against Oil, Lab, and Stand and against the predicted values of New_WPD, presented in Figures 34, 35, 36, 37, and 38, reveal nothing particularly troublesome. The residuals based on observations from labs D and F appear to have slightly lower variance than the residuals based on the observations from the other three labs. Nonetheless, there do not appear to be major departures from the assumption of an error term with constant variance.

Figure 34: Scatterplot of Residuals vs. Predicted Values for New_WPD

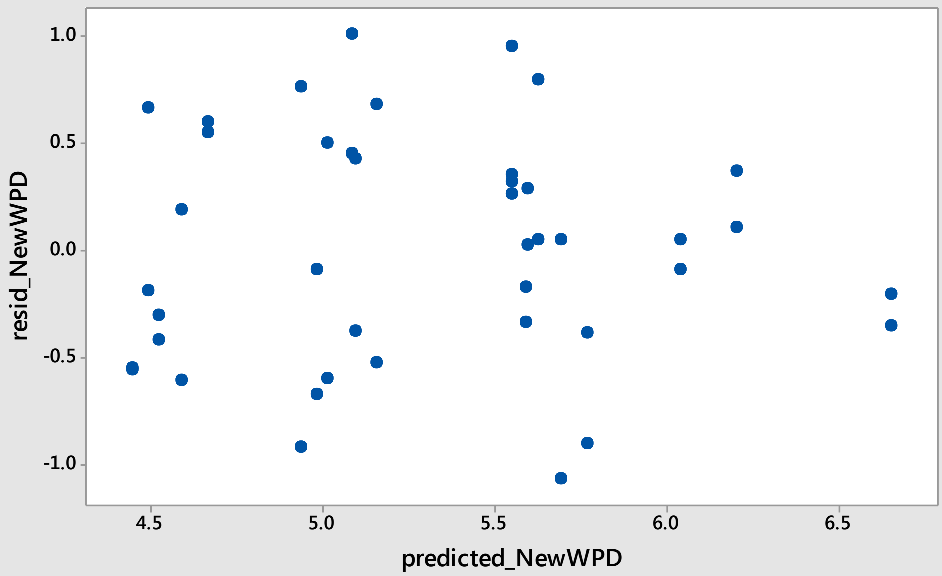


Figure 35: Scatterplot of Residuals vs. Oil for New_WPD

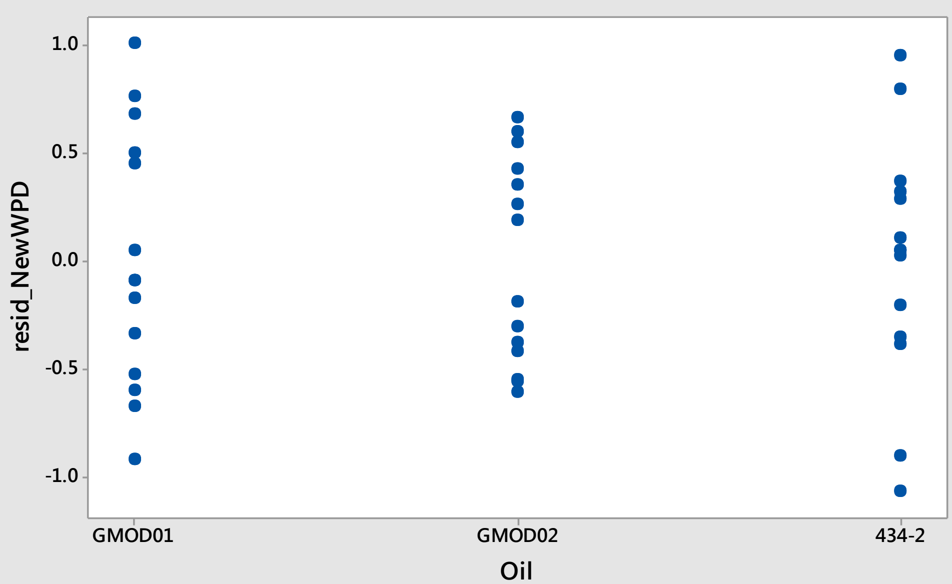


Figure 36: Scatterplot of Residuals vs. Lab for New_WPD

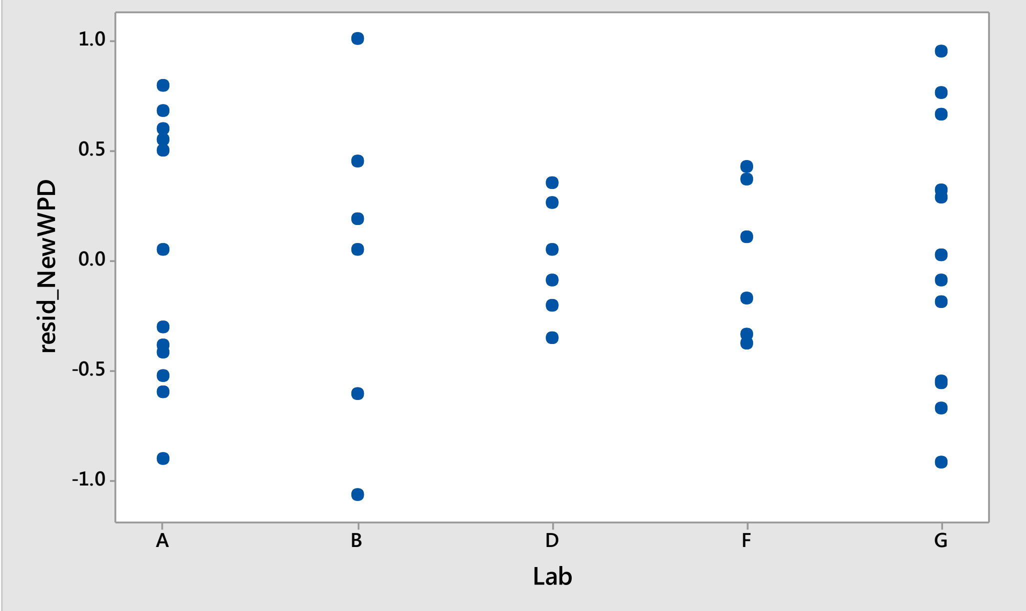
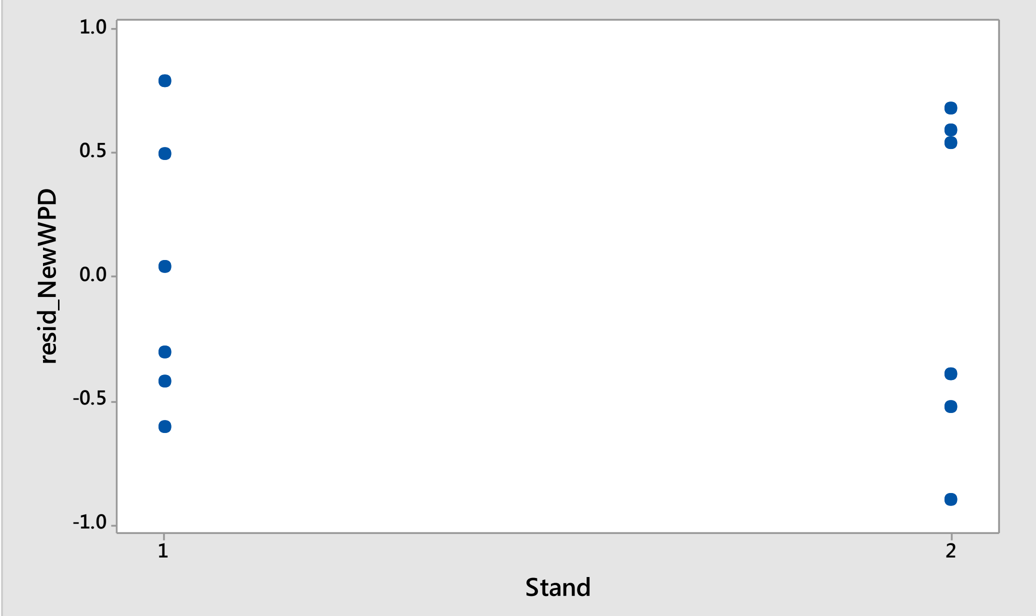


Figure 37: Scatterplot of Residuals vs. Stand for New_WPD, Lab A



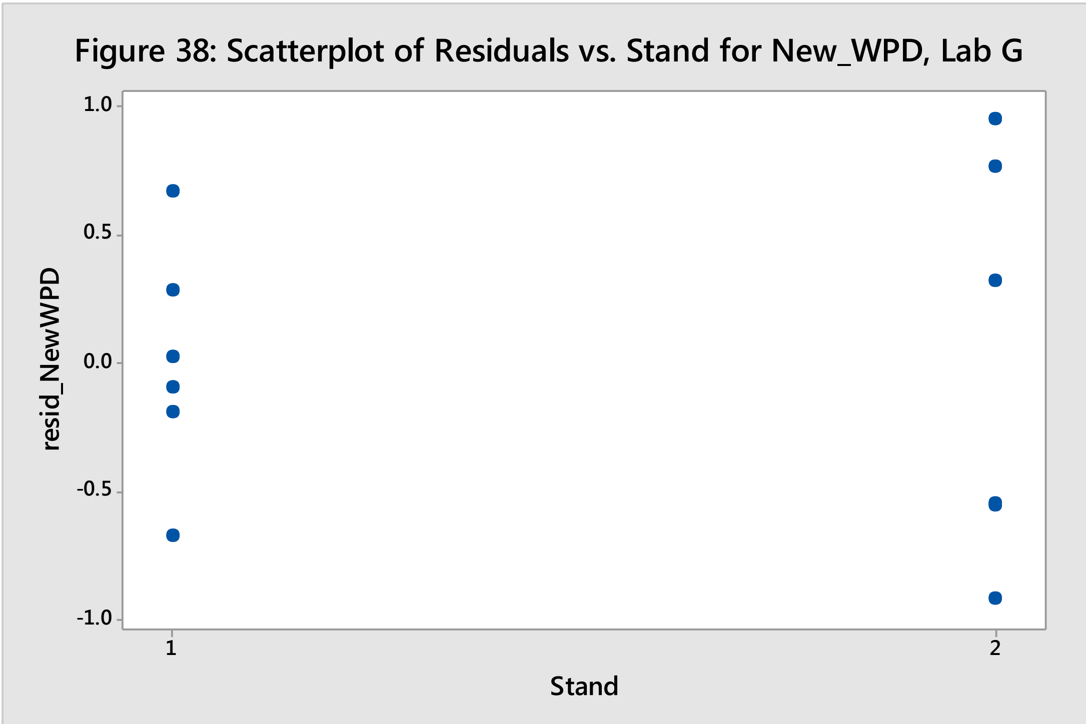


Figure 38: Scatterplot of Residuals vs. Stand for New_WPD, Lab G
1.0
0.5
0.0
-0.5
-1.0
resid_NewWPD
1
2
Stand